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# ON RANDOM FIXED POINT THEOREMS FOR EXPANSIVE TYPE MULTIVALUED OPERATOR IN POLISH SPACE

SARALA CHOUHAN & NEERAJ MALVIYA

**ABSTRACT :** The objective of this paper is to obtain some fixed point theorems for pair of non commuting expansive type multivalued operators on Polish space.

**Key words :** Polish space, random multivalued operator, random fixed point, Hausdroff metric, measurable mapping.
**Mathematics Subject Classification**. 47H10, 54H25

## 1. INTRODUCTION

Random fixed point theorems are stochastic generalization of classical fixed point theorems [7, 15]. Itoh [9, 10] extended several well known fixed point theorems, i.e., for contraction, nonexpansive and condemning, mappings to the random case. Thereafter, various stochastic aspects of Schauder's fixed point theorem have been studied by Sehgal and Singh [16], Papageorgiou [14], Lin [12] and many authors. In a separable metric space, random fixed point theorems for contractive mappings were proved by Spacek [15], Hans [6, 7, 8], Mukherjee [13]. Afterwards, Beg and Shahzad [2, 3], Badshah and Sayyed [4] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved the random fixed points theorems for contraction random operators in Polish space. In present paper random fixed point theorems for pair of non commuting expansive type mapping in Polish *space are investigated.*

## 2. PRELIMINARIES

Let $(X, d)$ be a polish space, that is, a separable complete metric space and $(\Omega, A)$ be measurable space. Let $2^X$ be a family of all subsets of $X$ and $CB(X)$ denote the family of all non-empty

bounded closed subsets of $X$. A mapping $T : \Omega \to 2^X$ is called measurable if for all open subsets $C$ of $X$, $T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in A$. A mapping $\xi : \Omega \to X$ is said to be measurable selector of a measurable mapping $T : \Omega \to 2^X$ If $\xi$ is measurable and $\xi(\omega) \in T(\omega)$ for all $\omega \in \Omega$. A mapping $f : \Omega \times X \to X$ is called a random operator if for all $x \in X$, $f(; x)$ is measurable. A mapping $T : \Omega \times X \to CB(X)$ is called a random multivalued operator, if for every $x \in X$, $T(; x)$ is meaurable. A measurable mapping $T : \Omega \times X$ is called random fixed point of a random multivalued operator $T : \Omega \times X \to CB(X)$ ($f : \Omega \times X \to X$), if for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))(f(\omega, \xi(\omega)) = \xi(\omega))$. Let $T : \Omega \times X \to CB(X)$ be a random operator and $\{\xi_n\}$ a sequence of measurable mappings $\xi_n : \Omega \to X$. The sequence $\{\xi_n\}$ is said to be asymptotically $T$-regular if $d(\xi_n(\omega), T(\omega, \xi_n(\omega))) \to 0$.

## 3. MAIN RESULTS

**Theorem 3.1.** Let $X$ be a polish space. Let $S, T : \Omega \times X \to CB(X)$ be two non commuting surjective random multivalued operators. If there exist measurable mappings $\alpha, \beta, \gamma : \Omega \to (0, 1)$ such that

$$H(ST(\omega, x), TS(\omega, y)) \geq \frac{a(\omega)\left[d(x, ST(\omega, x))\, d(x, y) + d(y, TS(\omega, y))\, d(x, y)\right]}{d(x, ST(\omega, x)) + d(y, TS(\omega, y)) + d(x, y)}$$

$$+\frac{\beta(\omega) d\left(x, ST(\omega, x)\right) d\left(y, TS(\omega, y)\right) + \gamma(\omega)\left[d(x, y)\right]^2}{d(x, ST(\omega, x)) + d(y, TS(\omega, y)) + d(x, y)} \qquad (3.1.1)$$

for each $x, y \in X$, $x \neq y$, $\omega \in \Omega$ where $\alpha, \beta, \gamma \in R^+$, with $2\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 3$ and $\gamma(\omega) > 1$. Then $ST$ and $TS$ have a common fixed point.

*(Here H represents the Hausdroff metric on CB(X) induced by the metric d)*

***Proof :*** We define a sequence $\{\xi_n(\omega)\}$ for each $\omega \in \Omega$ as follows for $n = 0, 1, 2$ .......

$$\xi_{2n}(\omega) \in ST(\omega, \xi_{2n+1}(\omega)), \qquad ...(3.1.2)$$

$$\xi_{2n+1}(\omega) \in TS(\omega, \xi_{2n+2}(\omega))$$

*Now consider*

$$d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) = H[ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, \xi_{2n+2}(\omega))]$$

$$\geq \frac{\alpha(\omega)\Big[d\big(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\big)\, d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}{d\big(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\big) + d\big(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$+ \frac{d\big(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))\big) d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)\Big]}{d\big(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\big) + d\big(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$+ \frac{\beta(\omega) d\big(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\big) d\big(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))\big)}{d\big(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\big) + d\big(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$+ \frac{\gamma(\omega)\Big[d^2\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)\Big]}{d\big(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\big) + d\big(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$= \frac{\alpha(\omega)\Big[d\big(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\big) d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}{d\big(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\big) + d\big(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$+ \frac{d\big(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)\big) d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)\Big]}{d\big(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\big) + d\big(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$+ \frac{\beta(\omega) d\big(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\big) d\big(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)\big)}{d\big(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\big) + d\big(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$+ \frac{\gamma(\omega) d^2\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}{d\big(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\big) + d\big(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)\big) + d\big(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\big)}$$

$$\Rightarrow d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))[d(\xi_{2n}(\omega), \xi_{2n+1}(\omega) + 2d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))]$$

$$\geq d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))[2\alpha(\omega) + \beta(\omega) + \gamma(\omega)]$$

$$\min \{d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\}$$

$\Rightarrow d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))[2\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 2]$
$\min \{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\}$

***Case I***

$$d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq [2\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 2]d^2(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega) \leq \left(\frac{1}{2\alpha(\omega)+\beta(\omega)+\gamma(\omega)-2}\right)^{1/2} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k_1(\omega)\mathrm{d}(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

*where* $k_1 = k_1(\omega) = \left(\frac{1}{2\alpha(\omega)+\beta(\omega)+\gamma(\omega)-2}\right)^{1/2} < 1$ [As $2\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 3$]

*Similarly we can calculate*

$$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k_1(\omega)\ d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

*where* $k_1 = k_1(\omega) = \left(\frac{1}{2\alpha(\omega)+\beta(\omega)+\gamma(\omega)-2}\right)^{1/2} < 1$ [As $2\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 3$]

*and so on*

***Case II***

$$d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq [2\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 2]d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\mathrm{d}(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq \frac{1}{2\alpha(\omega)+\beta(\omega)+\gamma(\omega)-2}\ d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k_2(\omega)\ d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

*where* $k_2 = k_2(\omega) = \frac{1}{2\alpha(\omega)+\beta(\omega)+\gamma(\omega)-2} < 1$ [As $2\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 3$]

*Similarly we can calculate*

$$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k_2(\omega)\ d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

*where* $k_2 = k_2(\omega) = \left(\frac{1}{2\alpha(\omega)+\beta(\omega)+\gamma(\omega)-2}\right) < 1$ [As $2\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 3$]

*and so on*

*So, in general*

$d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k\ d(\xi_{n-1}(\omega), \xi_n(\omega))$ *for* $n = 1, 2, 3.....$

*where* $k = k(\omega) = \max\{k_1(\omega), k_2(\omega)\}$ *then* $k < 1$

$\Rightarrow d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k^n\ d(\xi_0(\omega), \xi_1(\omega))$

Now we shall prove that for each $\omega \in \Omega$ $\{\xi_n(\omega)\}$ is a Cauchy sequence. For this for every positive integer p, wc have

$$d(\xi_n(\omega), \xi_{n+p}(\omega)) \leq d(\xi_n(\omega), \xi_{n+1}(\omega)) + d(\xi_{n+1}(\omega), \xi_{n+2}(\omega)) + \ldots + d(\xi_{n+p-1}(\omega), \xi_{n+p}(\omega))$$

$$\leq (k^n + k^{n+1} + k^{n+2} + \ldots.. + k^{n+p-1}) d(\xi_0(\omega), \xi_1(\omega))$$

$$= k^n(1 + k + k^2 + \ldots.. + k^{p-1})\ d(\xi_0(\omega), \xi_1(w))$$

$$< \frac{k^n}{(1-k)}(\xi_0(\omega), \xi_1(\omega))$$

*which tends to zero as* $n \to \infty$. *It follows that* $\{\xi_n(\omega)\}$ *is a Cauchy sequence and there exists a measurable mapping* $\xi : \Omega \to X$ *such that* $\xi_n(\omega) \to \xi(\omega)$ *for each* $\omega \in \Omega$ ... (3.1.3)

***Existence of random fixed point:*** *Since S and T are surjective maps so ST and TS are also surjective and hence there exist two functions* $g : \Omega \to X$ *and* $g' : \Omega \to X$ *such that.*

$$\xi(\omega) \in ST(\omega, g(\omega)) \text{ and } \xi(\omega) \in TS(\omega, g'(\omega)) \qquad \ldots(3.1.4)$$

*Consider*

$$d(\xi_{2n}(\omega), \xi(\omega)) = H(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, g'(\omega)))$$

$$\geq \frac{\alpha(\omega)\left[d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(\xi_{2n+1}(\omega), g'(\omega))\right.}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(g'(\omega), TS(\omega, g'(\omega))) + d(\xi_{2n+1}(\omega), g'(\omega))}$$

$$+ \frac{\left. d(g'(\omega), TS(\omega, g'(\omega)))d(\xi_{2n+1}(\omega), g'(\omega))\right]}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(g'(\omega), TS(\omega, g'(\omega))) + d(\xi_{2n+1}(\omega), g'(\omega))}$$

$$+ \frac{\beta(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(g'(\omega), TS(\omega, g'(\omega)))}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(g'(\omega), TS(\omega, g'(\omega))) + d(\xi_{2n+1}(\omega), g'(\omega))}$$

$$+\frac{\gamma(\omega)d^2\left(\xi_{2n+1}(\omega),\, g'(\omega)\right)}{d\left(\xi_{2n+1}(\omega),\, ST\left(\omega,\, \xi_{2n+1}(\omega)\right)\right)+d\left(g'(\omega),\, TS\left(\omega, g'(\omega)\right)\right)+d\left(\xi_{2n+1}(\omega),\, g'(\omega)\right)}$$

$$=\frac{\alpha(\omega)\left[d\left(\xi_{2n+1}(\omega),\, \xi_{2n}(\omega)\right)d\left(\xi_{2n+1}(\omega), g'(\omega)\right)+d\left(g'(\omega), \xi(\omega)\right)d\left(\xi_{2n+1}(\omega), g'(\omega)\right)\right]}{d\left(\xi_{2n+1}(\omega),\, \xi_{2n+1}(\omega)\right)+d\left(g'(\omega),\, \xi(\omega)\right)+d\left(\xi_{2n+1}(\omega),\, g'(\omega)\right)}$$

$$+\frac{\beta(\omega)d\left(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\right)d\left(g'(\omega), \xi(\omega)\right)}{d\left(\xi_{2n+1}(\omega),\, \xi_{2n}(\omega)\right)+d\left(g'(\omega),\, \xi(\omega)\right)+d\left(\xi_{2n+1}(\omega),\, g'(\omega)\right)}$$

$$+\frac{\gamma(\omega)\left[d\left(\xi_{2n+1}(\omega), g'(\omega)\right)\right]^2}{d\left(\xi_{2n+1}(\omega),\, \xi_{2n}(\omega)\right)+d\left(g'(\omega),\, \xi(\omega)\right)+d\left(\xi_{2n+1}(\omega),\, g'(\omega)\right)}$$

*As* $\{\xi_{2n}(\omega)\}$ *and* $\{\xi_{2n+1}(\omega)\}$ *are subsequences of* $\{\xi_n(\omega)\}$ *as* $n \to \infty$, $\{\xi_{2n}(\omega)\} \to \xi(\omega)$ $\{\xi_{2n+1}(\omega)\} \to \xi(\omega)$ (*using* 3.1.3)

*Therefore*

$$d(\xi(\omega),\, \xi(\omega)) \geq \frac{\alpha(\omega)\left[d\left(\xi(\omega), \xi(\omega)\right)d\left(\xi(\omega), g'(\omega)\right)+d\left(g'(\omega), \xi(\omega)\right)d\left(\xi(\omega), g'(\omega)\right)\right]}{d\left(\xi(\omega), \xi(\omega)\right)+d\left(g'(\omega), \xi(\omega)\right)+d\left(\xi(\omega), g'(\omega)\right)}$$

$$+\frac{\beta(\omega)d\left(\xi(\omega)\right)d\left(g'(\omega), \xi(\omega)\right)+\gamma(\omega)d^2\left(\xi(\omega), g'(\omega)\right)}{d\left(\xi(\omega), \xi(\omega)\right)+d\left(g'(\omega), \xi(\omega)\right)+d\left(\xi(\omega), g'(\omega)\right)}$$

$$0 \geq \frac{\left(\alpha(\omega)+\gamma(\omega)\right)}{2}\; d(g'(\omega),\, \xi(\omega))$$

$$\Rightarrow d(g'(\omega),\, \xi(\omega)) = 0 \qquad (as\ \alpha(\omega) + \gamma(\omega) > 1)$$

$$\Rightarrow \xi(\omega) = g'(\omega) \tag{3.1.5}$$

*In an exactly similar way we can prove that*

$$\Rightarrow \xi(\omega) = g(\omega) \tag{3.1.6}$$

The fact (3.1.4) along with (3.1.5) and (3.1.6) show that $\xi(\omega)$ is a common fixed point of $ST$ and $TS$.

This completes the proof of the theorem 3.1.

**Theorem 3.2.** Let $X$ be a polish space. Let $S, T : \Omega \times X \to CB(X)$ be two non commuting continuous surjective random multivalued operators. It there exist measurable mappings $\alpha$, $\beta$, $\gamma : \Omega \to (0, 1)$ such that

$$H(ST(\omega, x), TS(\omega, y)) \geq \frac{\alpha(\omega)d(x, ST(\omega,x))d(y, TS(\omega,y))}{d(x,y)} + \beta(\omega)[d(x, ST(\omega, x))$$

$$+ d(y, TS(\omega, y))] + \gamma(\omega)d(x, y) \qquad \text{...(3 1.7)}$$

for each $x, y \in X$, $x \neq y$ $\omega \in \Omega$ where $\alpha, \beta, \gamma \in R^+$, with $\alpha(\omega) + 2\beta(\omega) + \gamma(\omega) > 1$. Then $ST$ and $TS$ have a common fixed point.

***Proof.*** We define a sequence $\{\xi_n(\omega)\}$ for each $\omega \in \Omega$ as follows for $n = 0, 1, 2.....$

$$\xi_{2n}(\omega) \in ST(\omega, \xi_{2n+1}(\omega))$$

$$\xi_{2n+1}(\omega) \in TS(\omega, \xi_{2n+2}(\omega)) \qquad \text{...(3.1.8)}$$

*Now consider*

$$d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) = H[ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, \xi_{2n+2}(\omega))]$$

$$\geq \frac{\alpha(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))}{d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}$$

$$+ \beta(\omega)[d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))]$$

$$+ \gamma(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

$$= \frac{\alpha(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))}{d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}$$

$$+ \beta(\omega)[d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))] + \gamma(\omega)\mathrm{d}(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

$= (\alpha(\omega) + \beta(\omega))d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) + (\beta(\omega) + \gamma(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$

$\Rightarrow [1 - (\alpha(\omega) + \beta(\omega))]\, d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq (\beta(\omega) + \gamma(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq \frac{[1 - (\alpha(\omega) + \beta(\omega))]}{\beta(\omega) + \gamma(\omega)}\, d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k(\omega)d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$

$$\textit{where } k = k(\omega) = \left[\frac{1 - (\alpha(\omega) + \beta(\omega))}{\beta(\omega) + \gamma(\omega)}\right] < 1 \text{ [As } \alpha(\omega) + 2\beta(\omega) + \gamma(\omega) > 1]$$

*Similarly we can calculate*

$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$

$$\textit{where } k = k(\omega) = \left[\frac{1 - (\alpha(\omega) + \beta(\omega))}{\beta(\omega) + \gamma(\omega)}\right] < 1 \text{ [As } \alpha(\omega) + 2\beta(\omega) + \gamma(\omega) > 1]$$

*and so on*

*So, in general*

$d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k\, d(\xi_{n-1}(\omega), \xi_n(\omega))$ *for* $n = 1, 2, \ldots$

$\Rightarrow d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k^n d(\xi_0(\omega), \xi_1(\omega))$ ...(3.1.9)

Now we can prove that for each $\omega \in \Omega$ $\{\xi_n(\omega)\}$ is a Cauchy sequence. (As proved in theorem 3.1) So there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. ...(3.1.10)

***Existence of random fixed point :*** *Since S and T are surjective maps so ST and TS are also surjective and hence there exist two functions* $g : \Omega \to X$ *and* $g' \cdot \Omega \to X$ *such that.*

$\xi(\omega) \in ST(\omega, g(\omega))$ *and* $\xi(\omega)\ TS(\omega, g'(\omega))$ ...(3.1.11)

*Consider*

$d(\xi_{2n}(\omega), \xi(\omega)) = H(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, g'(\omega)))$

$$\geq \frac{\alpha(\omega)d\left(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))\right)d\left(g'(\omega), TS(\omega, g'(\omega))\right)}{d\left(\xi_{2n+1}(\omega), g'(\omega)\right)}$$

$$+ \beta(\omega)[d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(g'(\omega), TS(\omega, g'(\omega)))] + \gamma(\omega)d(\xi_{2n+1}(\omega), g'(\omega))$$

$$= \frac{\alpha(\omega)d\left(\xi_{2n+1}(\omega), \xi_{2n}(\omega)\right)d\left(g'(\omega), \xi(\omega)\right)}{d\left(\xi_{2n+1}(\omega), g'(\omega)\right)}$$

$$+ \beta(\omega)[d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(g'(\omega), \xi(\omega))] + \gamma(\omega)d(\xi_{2n+1}(\omega), g'(\omega))$$

*As* $\{\xi_{2n}(\omega)\}$ *and* $\{\xi_{2n+1}(\omega)\}$ *are subsequences of* $\{\xi_n(\omega)\}$ *as* $n \to \infty$, $\{\xi_{2n}(\omega)\} \to \xi(\omega)$, $(\xi_{2n+1}(\omega)\} \to \xi(\omega)$

*Therefore*

$$d(\xi(\omega), \xi(\omega)) \geq \frac{\alpha(\omega)d\left(\xi(\omega)\right)d\left(g'(\omega), \xi(\omega)\right)}{d\left(\xi(\omega), g'(\omega)\right)} + \beta(\omega)[d(\xi(\omega), \xi(\omega)) + d(g'(\omega), \xi(\omega))]$$

$$+ \gamma(\omega)d(\xi(\omega), g'(\omega))$$

$$0 \geq (\beta(\omega) + \gamma(\omega))d(\xi(\omega), g'(\omega))$$

$$\Rightarrow d(\xi(\omega), g'(\omega)) = 0 \quad [As\ \beta(\omega) + \gamma(\omega) > 0]$$

$$\Rightarrow \xi(\omega) = g'(\omega) \qquad ...(3.1.12)$$

*In an exactly similar way we can prove that*

$$\Rightarrow \xi(\omega) = g'(\omega) \qquad ...(3.1.13)$$

The fact (3.1.11) along with (3.1.12) and (3.1.13) show that $\xi(\omega)$ is a common fixed point of $ST$ and $TS$.

This completes the proof of the theorem 3.2.

**Theorem 3.3.** Let $X$ be a polish space. Let $S, T : \Omega \times X \to CB(X)$ be two non commuting continuous surjective random multivalued operators. If there exist measurable mappings $\alpha$, $\beta$, $\gamma$, $K : \Omega \to (0, 1)$ such that

$$d(ST(\omega, x), TS(\omega, y)) + K(\omega)[d(x, TS(\omega, y)) + d(y, ST(\omega, x))] \geq \alpha(\omega)d(x, ST(\omega, x)) + \beta(\omega)d(y, TS(\omega, y)) + \gamma(\omega)d(x,y) \quad ...(3.1.14)$$

for each $x, y \in X, x \neq y$ where $\alpha, \beta, \gamma, K \in R^+$ and $\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 1 + 2K(\omega)$, $\beta(\omega) + \gamma(\omega) > K(\omega)$, $\alpha(\omega) + \gamma(\omega) > K(\omega)$ and $\gamma(\omega) > 2K(\omega)$. Then $ST$ and $TS$ have a common fixed point.

**Proof.** We define a sequence $\{\xi_n(\omega)\}$ for each $\omega \in \Omega$ as follows for $n = 0, 1, 2$...

$$\xi_{2n}(\omega) \in ST(\omega, \xi_{2n+1}(\omega))$$

$$\xi_{2n+1}(\omega) \in TS(\omega, \xi_{2n+2}(\omega)) \quad ...(3.1.15)$$

*Now we put* $x = \xi_{2n+1}(\omega)$ *and* $y = \xi_{2n+2}(\omega)$ *in* (3.1.14) *we get*

$$d(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, \xi_{2n+2}(\omega))) + K(\omega)[d(\xi_{2n+1}(\omega), TS(\omega, \xi_{2n+2}(\omega))) + d(\xi_{2n+2}(\omega), ST(\omega, \xi_{2n+1}(\omega)))]$$

$$\geq \alpha(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + \beta(\omega)d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))) + \gamma(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

$$\Rightarrow d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) + K(\omega)[d(\xi_{2n+2}(\omega), \xi_{2n}(\omega))]$$

$$\geq \alpha(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + \beta(\omega)d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)) + \gamma(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

$$\Rightarrow (1 + K(\omega) - \alpha(\omega))d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq (\beta(\omega) + \gamma(\omega) - K(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega) \leq \frac{(1 + K(\omega) - \alpha(\omega))}{\beta(\omega) + \gamma(\omega) - K(\omega)} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k(\omega)d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

$$\text{where } k = k(\omega) = \frac{[1 + K(\omega) - \alpha(\omega)]}{\beta(\omega) + \gamma(\omega) - K(\omega)} < 1$$

[Since $\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 1 + 2K(\omega)$]

*Similarly we can calculate*

$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$ *for* $n = 0, 1, 2, 3 \ldots$

*where* $k = k(\omega) = \dfrac{[1 + K(\omega) - \alpha(\omega)]}{\beta(\omega) + \gamma(\omega) - K(\omega)} < 1$

*and so on*

*So, in general*

$d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k\ d(\xi_{n-1}(\omega), {}_n(\omega))$ *for* $n = 1, 2, 3 \ldots$

$$\Rightarrow d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k^n\ d(\xi_0(\omega), \xi_1(\omega)) \qquad \ldots(3.1.16)$$

Now we can prove that for each $\omega \in \Omega$ $\{\xi_n(\omega)\}$ is a Cauchy sequence. (As proved in theorem 3.1) So there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$.

***Existence of random fixed point :-*** *Since S and T are surjective maps so ST and TS are also surjective and hence there exist two functions* $g : \Omega \to X$ *and* $g' : \Omega \to X$ *such that.*

$$\xi(\omega) \in ST(\omega, g(\omega)) \text{ and } \xi(\omega) \in TS(\omega, g'(\omega)) \qquad \ldots(3.1.17)$$

*Consider*

$d(\xi_{2n}(\omega), \xi(\omega)) = H(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, g'(\omega)))$

$\geq -K(\omega)[d(\xi_{2n+1}(\omega), TS(\omega, g'(\omega)) + d(g'(\omega), ST(\omega, \xi_{2n+1}(\omega)))] + \alpha(\omega)d(\xi_{2n+1}(\omega),$

$ST(\omega, \xi_{2n+1}(\omega))) + \beta(\omega)d(g'(\omega), TS(\omega, g'(\omega))) + \gamma(\omega)d(\xi_{2n+1}(\omega), g'(\omega))$

$\Rightarrow d(\xi_{2n}(\omega), TS(\omega, g'(\omega)) \geq -K(\omega)[d(\xi_{2n+1}(\omega), \xi(\omega)) + d(g'(\omega), \xi_{2n}(\omega))]$

$+ \alpha(\omega)d(\xi_{2n+1}(\omega), {}_{2n}(\omega)) + \beta(\omega)d(g'(\omega), \xi(\omega)) + \xi(\omega)d(\xi_{2n+1}(\omega), g'(\omega))$

*As* $\{\xi_{2n}(\omega))$ *and* $\{\xi_{2n+1}(\omega)\}$ *are subsequences of* $\{\xi_n(\omega)\}$ *as* $n \to \infty$, $\{\xi_{2n}(\omega)\} \to \xi(\omega)$ $\{\xi_{2n+1}(\omega)\} \to \xi(\omega)$

*Therefore*

$$\Rightarrow d(\xi(\omega), \xi(\omega)) \geq -K(\omega)[d(\xi(\omega), \xi(\omega)) + d(g'(\omega), \xi(\omega))] + \alpha(\omega)d(\xi(\omega), \xi(\omega))$$
$$+ \beta(\omega)d(g'(\omega), \xi(\omega)) + \gamma(\omega)d(\xi(\omega), g'(\omega))$$

$$\Rightarrow 0 \geq (\beta(\omega) + \gamma(\omega) - K(\omega))d(\xi(\omega), g'(\omega))$$

$$\Rightarrow d(\xi(\omega), g'(\omega)) = 0 \quad [\text{As } \beta(\omega) + \gamma(\omega) - K(\omega) > 0]$$

$$\Rightarrow \xi(\omega) = g'(\omega) \qquad \ldots (3.1.18)$$

*In an exactly similar way (using* $\alpha(\omega) + \gamma(\omega) > K(\omega)$*) we can prove that*

$$\xi(\omega) = g(\omega) \qquad \ldots(3.1.19)$$

The fact (3.1.17) along with (3.1.18) and (3.1.19) show that $\xi(\omega)$ is a common fixed point of *ST* and *TS*.

This completes the proof of the theorem 3.3.

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# GENERALIZED SEMI-PRE HOMEOMORPHISMS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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**ABSTRACT :** In this paper we introduce the new class of homeomorphisms called generalized semi-pre homeomorphisms in intuitionistic fuzzy topological spaces. We also introduce M-generalized semi-pre homeomorphisms in intuitionistic fuzzy topological spaces and investigate some of the properties We provide the relation between intuitionistic fuzzy generalized semi-pre homeomorphisms and intuitionistic fuzzy M-generalized semi-pre homeomorphisms. Also we prove that the set of all intuitionistic fuzzy M-generalized semi-pre homeomorphisms forms a group under the operation of composition of maps.

**Key words and phrases :** Intuitionistic fuzzy topology, intuitionistic fuzzy generalized semi-pre $T_{1/2}$ space, intuitionistic fuzzy generalized semi-pre homeomorphisms and intuitionistic fuzzy M-generalized semi-pre homeomorphisms.

## 1. INTRODUCTION

After the introduction of fuzzy sets by Zadeh [9], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Using the notion of intuitionistic fuzzy sets, Coker [3] introduced the notion of intuitionistic fuzzy topological spaces. The notion of homeomorphisms plays a vital role in intuitionistic fuzzy topology as well as in topology. Here we introduce the new class of homeomorphisms called generalized semi-pre homeomorphisms in intuitionistic fuzzy topological spaces. We also introduce the M-generalized semi-pre homeomorphisms in intuitionistic fuzzy topological spaces and investigate some of the properties. We provide the relation between intuitionistic fuzzy generalized semi-pre homeomorphisms and intuitionistic fuzzy M-generalized semi-pre homeomorphisms. Also we prove that the set of all intuitionistic fuzzy M-generalized semi-pre homeomorphisms forms a group under the operation of composition of maps.

## 2. PRELIMINARIES

**Definition 2.1:** [1] An *intuitionistic fuzzy set* (IFS in short) A in X is an object having the form

$$A = \{(x, \mu_A(x), \nu_A(X)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS $(X)$, the set of all intuitionistic fuzzy sets in $X$.

**Definition 2.2:** [1] Let $A$ and $B$ be IFSs of the form $A = \{(x, \mu_A(x), \nu_A(x)) / x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) / x \in X\}$.

Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A'$

(c) $A^c = \{(x, \nu_A(x), \mu_A(x)) / x \in X\}$

(d) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee {}_B(x)) / x \in X\}$

(e) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) / x \in X\}$

For the sake of simplicity, we shall use the notation $A = (x, \mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x))/x \in X\}$.

The intuitionistic fuzzy sets $0_\sim = \{(x, 0, 1)/x \in X)$ and $1_\sim = \{(x, 1, 0)/x \in X\}$ are respectively the empty set and the whole set of $X$.

**Definition 2.3:** [3] An *intuitionistic fuzzy topology* (IFT for short) on $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms.

(i) $0_\sim, 1_\sim \in \tau$

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(iii) $\cup G_i \in \tau$ for any family $\{G_i / i \in j\} \subseteq \tau$.

In this case the pair $(X, \tau)$ is called an *intuitionistic fuzzy topological space* (IFTS in short) and any IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS in short) in $X$. The complement $A^c$ of an IFOS $A$ in IFTS $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS in short) in $X$.

**Definition 2.4:** [4] Let $(X, \tau)$ be an IFTS and $A = (x, \mu_A, \nu_A)$ be an IFS in $X$. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure are defined by

$\text{int}(A) = \cup \{G/G$ is an IFOS in $X$ and $G \subseteq A\}$

$\text{cl}(A) = \cap \{K/K$ is an IFCS in $X$ and $A \subseteq K\}$

**Definition 2.5:** [4] An IFS $A = (x, \mu_A, \nu_A)$ in an IFTS $(X, \tau)$ is said to be an *intuitionistic fuzzy pre closed set* (IFPCS in short) if $\text{cl}(\text{int}(A)) \subseteq A$ and *intuitionistic fuzzy pre open set* (IFPOS in short) if $A \subseteq \text{int}(\text{cl}(A))$.

**Definition 2.6:** [8] An IFX A = $(x, \mu_A, \nu_A)$ in an IFTS $(X, \tau)$ is said to be an

(i) *intuitionistic fuzzy semi-pre closed set* (IFSPCS for short) if there exists an IFPCS $B$ such that $\text{int}(B) \subseteq A \subseteq B$.

(ii) *intuitionistic fuzzy semi-pre open set* (IFSPOS for short) if there exists an intuitionistic fuzzy pre open set (IFPOS for short) $B$ such that $B \subseteq A \subseteq cl(B)$.

**Definition 2.7:** [5] Let $A$ be an IFS in an IFTS $(X, \tau)$. Then the semi-pre interior and the semi-pre closure of $A$ are defined by

$\text{spint}(A) = \cup\{G/G$ is an IFSPOS in $X$ and $G \subseteq A\}$.

$\text{spcl}(A) = \cap \{K/K$ is an IFSPCS in $X$ and $A \subseteq K\}$.

Note that for any IFS $A$ in $(X, \tau)$, we have $\text{spcl}(A^c) = [\text{spint}(A)]^c$ and $\text{spint}(A^c) = [\text{spcl}(A)]^c$ [5].

**Definition 2.8:** [8] An IFS $A$ in an IFTS $(X, \tau)$ is said to be an *intuitionistic fuzzy generalized semi-pre closed set* (IFGSPCS for short) if $\text{spcl}(A) \subseteq U$ whernever $A \subseteq U$ and $U$ is an IFOS in $(X, \tau)$.

Every IFSPCS is an IFGSPCS but the converse may not be true in general [8].

**Definition 2.9:** [5] The complement $A^c$ of an IFGSPCS $A$ in an IFTS $(X, \tau)$ is called an *intuitionistic fuzzy generalized semi-pre open set* (IFGSPOS for short) in $X$.

**Definition 2.10:** [5] If every IFGSPCS in $(X, \tau)$ is an IFSPCS in $(X, \tau)$, then the space can be called as an *intuitionistic fuzzy semi-pre $\mathcal{T}_{1/2}$ space* (IFSPT$_{1/2}$ space for short).

**Definition 2.11:** [6] A mapping $f: (X, \tau) \to (Y, \sigma)$ is called an *intuitionistic fuzzy generalized semi-pre continuous mapping* (IFGSP continuous mapping for short) if $f^{-1}(V)$ is an IFGSPCS in $(X, \tau)$ for every IFCS $V$ of $(Y, \sigma)$.

**Definition 2.12:** [6] A mapping $f: (X, \tau) \to (Y, \sigma)$ is called *intuitionistic fuzzy generalized semi-pre irresolute* (IFGSP irresolute) mapping if $f^{-1}(V)$ is an IFGSPCS (IFGSPOS) in $(X, \tau)$ for every IFGSPCS (IFGSPOS) $V$ of $(Y, \sigma)$.

**Definition 2.13:** [7] A map $f: X \to Y$ is called an *intuitionistic fuzzy generalized semi-pre closed mapping* (IFGSPCM for short) if $f(A)$ is an IFGSPCS in $Y$ for each IFCS $A$ in $X$.

**Definition 2.14:** [7] A mapping $f: X \to Y$ is said to be an *intuitionistic fuzzy generalized semi-pre open mapping* (IFGSPOM for short) if $f(A)$ is an IFGSPOS in $Y$ for each IFOS in $X$.

**Definition 2.15:** [7] A mapping $f: X \to Y$ is said to be an *intuitionistic fuzzy $\mathcal{M}$-generalized semi-pre closed mapping* (IFMGSPCM, for short) (if $f(A)$ is an IFGSPCS in $Y$ for every IFGSPCS $A$ in $X$.

## 3. GENERALIZED SEMI-PRE HOMEOMORPHISMS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

In this section we introduce intuitionistic fuzzy generalized semi-pre homeomorphisms and investigate some of their properties.

**Definition 3.1:** Let $f: X \to Y$ be a bijective mapping. Then $f$ is said to be an intuitionistic

fuzzy generalized semi-pre homeomorphism (IFGSPHM for short) if $f$ is both an IFGSP continuous mapping and an IFGSPOM.

For the sake of simplicity, we shall use the notation $A = (x, (\mu_a, \mu_b), (\nu_a, \nu_b)$ instead of $A = \left(x, \left(\frac{a}{\mu_a}, \frac{b}{\mu_b}\right), \left(\frac{a}{\nu_a}, \frac{b}{\nu_b}\right)\right)$ in the following examples.

Similarly we shall use the notation $B = (y, (\mu_u, \mu_v), (\nu_u, \nu_v))$ instead of $B = \left(y, \left(\frac{u}{\mu_u}, \frac{v}{\mu_v}\right), \left(\frac{u}{\nu_u}, \frac{v}{\nu_v}\right)\right)$ in the following examples.

**Example 3.2:** Let $X = \{a, b\}$, $Y = \{u, v\}$ and $G_1 = (x, (0.5_a, 0.6_b), (0.5_a, 0.4_b))$, $G_2 = (y, (0.2_u, 0.3_v), (0.8_u, 0.7_v))$. Then $\tau = \{0_\sim, G_1, 1_\sim\}$ and $\sigma = \{0_\sim, G_2, 1_\sim\}$ are IFTs on $X$ and $Y$ respectively. Define a mapping $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Then $f$ is an IFGSPHM.

**Theorem 3.3:** Let $f : X \to Y$ be a bijective mapping. If $f$ is an IFGSP continuous mapping, then the following are equivalent.

(i) $f$ is an IFGSPOM (ii) $f$ is an IFGSPHM (iii) $f$ is an IFGSPCM.

**Proof:** Straightforward.

**Remark 3.4:** The composition of two IFGSPHMs need not be an IFGSPHM in general.

**Example 3.5:** Let $X = \{a, b\}$, $Y = \{c, d\}$ and $Z = \{e, f\}$. Let $G_1 = (x, (0.5_a, 0.6_b), (0.5_a, 0.4_b))$, $G_2 = (x, (0.8_a, 0.7_b), (0.2_a, 0.3_b))$, $G_3 = (y, (0.8_c, 0.9_d), (0.2_c, 0.1_d))$, $G_4 = (z, (04_e, 0.3_f), (0.6_e, 0.7_f))$ and $G_5 = (z, (0.2_e, 0.2_f), (0.8_e, 0.8_f))$ and Then $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$, $\sigma = \{0_\sim, G_3, 1_\sim\}$ and $\eta = (0_\sim, G_4, G_5, 1\}$ are IFTs on $X$, $Y$ and $Z$ respectively. Define a mapping $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = c$ and $f(b) = d$ and $g : (Y, \sigma) \to (Z, \eta)$ by $g(c) = d$ and $g(d) = e$. Then $f$ and $g$ are IFGSPHMs but $g \circ f : X \to Z$ is not an IFGSPHM, since $g \circ f$ is not an IFGSP continuous mapping, since $G_4^c = (z, (0.6_e, 0.7_f), (0.4_e, 0.3_f))$ is an IFCS in $Z$ but $(g \circ f)^{-1}(G_4^c) = (x, (0.6_a, 0.7_b), (0.4_a, 0.3_b))$ is not an IFGSPCS in $X$, since $(g \circ f)^{-1}(G_4^c) = (x, (0.6_a, 0.7_b), (0.4_a, 0.3_b)) \subseteq G_2$ but $\mathrm{spcl}((g \circ f)^{-1}(G_4^c)) = 1_\sim \nsubseteq G_2$.

**Definition 3.6 :** Let $f: X \to Y$ be a bijective mapping. Then $f$ is said to be an intuitionistic fuzzy $M$-generalized semi-pre homeomorphism (IFMGSPHM for short) if $f$ is both an IFGSP irresolute mapping and an IFMGSPOM.

The family of all IFMGSPHMs in $X$ is denoted by IFMGSPHM($X$).

**Theorem 3.7:** Every IFMGSPHM is an IFGSPHM but not conversely.

**Proof :** Let $f: X \to Y$ be an IFMGSPHM. Let $A \subseteq Y$ be an IFCS. Then $A$ is an IFGSPCS in $Y$. By hypothesis, $f^{-1}(A)$ is an IFGSPCS in $X$. Hence $f$ is an IFGSP continuous mapping. Let $B \subseteq X$ be an IFOS. Then $B$ is an IFGSPOS in $X$. By hypothesis, $f(B)$ is an IFGSPOS in $Y$. Hence $f$ is an IFGSPOM. Thus $f$ is an IFGSPHM.

**Example 3.8:** Let $X = \{a, b\}$, $Y = \{u, v\}$ and $G_1 = (x, (0.4_a, 0.6_b), (0.6_a, 0.4_b))$, $G_2 = (x, (0.5_a, 0.7_b), (0.5_a, 0.3_b))$, $G_3 = (y, (0.2_u, 0.3_v), (0.8_u, 0.7_v))$, then $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ and $\sigma = \{0_\sim, G_3, 1_\sim\}$ are IFTs on $X$ and $Y$ respectively. Define a mapping $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = u$ and $f(b) = v$. Then $f$ is an IFGSPHM but not an IFMGSPHM, since $A = (y, (0.4_u, 0.7_v), (0.6_u, 0.3_v))$ is an IFGSPCS in $Y$ but $f^{-1}(A)$ is not an IFGSPCS in $X$, since $f^{-1}(A) = (x, (0.4_a, 0.7_b), (0.6_a, 0.3_b)) \subseteq G_2$ but $\text{spcl}(f^{-1}(A)) = 1_\sim \not\subseteq G_2$.

**Theorem 3.9 :** The composition of two IFMGSPHMs is an IFMGSPHM.

**Proof:** Let $f: X \to Y$ and $g: Y \to Z$ be any two IFMGSPHMs. Let $A \subseteq Z$ be an IFGSPCS. Then by hypothesis, $g^{-1}(A)$ is an IFGSPCS in $Y$. Again by hypothesis, $f^{-1}(g^{-1}(A))$ is an IFGSPCS in $X$. Therefore $g \circ f$ is an IFGSP irresolute mapping. Now let $B \subseteq X$ be an IFGSPCS. Then by hypothesis, $f(B)$ is an IFGSPOS in $Y$ and also $g(f(B))$ is an IFGSPOS in $Z$. This implies $g \circ f$ is an IFMGSPOM. Hence $g \circ f$ is an IFMGSPHM.

**Theorem 3.10:** Let $f: X \to Y$ be a bijective mapping. If $f$ is an IFGSP irresolute mapping, then the following are equivalent.

(i) $f$ is an IFMGSPOM (ii) $f$ is an IFMGSPHM (iii) $f$ is an IFMGSPCM

**Proof:** Straightforward.

**Theorem 3.11:** The set of all IFMGSPHMs in an IFTS $(X, \tau)$ is a group under the composition of maps.

**Proof:** Define a binary operation $*$ : IFMGSPHM($X$) $\times$ IFMGSPHM($X$) $\rightarrow$ IFMGSPHM($X$) by $f * g = g \text{ o } f$ for every $f, g \in$ IFMGSPHM($X$) and o is the usual operation of composition of maps. Since $g \in$ IFMGSPHM($X$) and $f \in$ IFMGSPHM($X$), by Theorem 3.9, $g \text{ o } f \in$ IFMGSPHM($X$). We know that the composition of maps is associative. The identity map $I : (X, \tau) \rightarrow (X, \tau)$ belonging to IFMGSPHM($X$) is the identity element. If $f \in$ IFMGSPHM($X$), then $f^{-1} \in$ IFMGSPHM($X$). Since if $A$ is an IFGSPOS in $X$, then $(f^{-1})^{-1}(A) = f(A)$ is an IFGSPOS in $Y$, by hypothesis that $f$ is an IFGSPOM. Therefore $f^{-1}$ is an IFGSP irresolute mapping. Similarly if $A$ is an IFGSPOS in $Y$, then $f^{-1}(A)$ in $X$ is an IFGSPOS, by the hypothesis that $f$ is an IFGSP irresolute mapping. Therefore $f^{-1}$ is an IFGSPOM. Hence $f^{-1}$ is an IFMGSPHM. Thus $f \text{ o } f^{-1} = f^{-1} \text{ o } f = \text{I}$ and so the inverse exists for each element of IFMGSPHM($X$). Hence (IFMGSPHM($X$), o) is a group under the composition of maps.

**Theorem 3.12:** Let $f: X \rightarrow Y$ be an IFMGSPHM. Then $f$ induces an isomorphism from the group IFMGSPHM($X$) onto the group IFMGSPHM($Y$).

**Proof:** Using $f$, we define a map $\varphi_f : h(X) \rightarrow h(Y)$ by $\varphi_f(h) = f \text{ o } h \text{ o } f^{-1}$ for every $h \in$ IFMGSPHM($X$). Then $\varphi_f$ is a bijection. Also for all $h_1, h_2 \in$ IFMGSPHM($X$), $\varphi_f(h_1 \text{ o } h_2) = f \text{ o } (h_1 \text{ o } h_2) \text{ o } f^{-1} = (f \text{ o } h_1 \text{ o } f^{-1}) \text{ o } (f \text{ o } h_2 \text{ o } f^{-1}) = \varphi_f(h_1) \text{ o } \varphi_f(h_2)$. This implies $\varphi_f$ is a homomorphism and so $\varphi_f$ is an isomorphism induced by $f$.

**Theorem 3.13:** If $f : X \rightarrow Y$ is an IFMGSPHM, then $\text{gspcl}(f^{-1}(B)) \subseteq f^{-1}(\text{spcl}(B))$ for every IFS $B$ in $Y$.

**Proof:** Let $B \subseteq Y$. Then spcl($B$) is an IFGSPCS in $Y$. Since $f$ is an IFGSP irresolute mapping, $f^{-1}(\text{spcl}(B))$ is an IFGSPCS in $X$. This implies $\text{gspcl}(f^{-1}(\text{spcl}(B))) = f^{-1}(\text{spcl}(B))$. Now $\text{gspcl}(f^{-1}(B)) \subseteq \text{gspcl}(f^{-1}(\text{spcl}(B)) = f^{-1}(\text{spcl}(B))$.

**Theorem 3.14:** If $f : X \rightarrow Y$ is an IFMGSPHM, where $X$ and $Y$ are IFSPT$_{1/2}$ spaces, then $\text{spcl}(f^{-1}(B)) = f^{-1}(\text{spcl}(B))$ for every IFS $B$ in $Y$.

**Proof:** Since $f$ is an IFMGSPHM, $f$ is an IFGSP irresolute mapping. Since $\mathrm{spcl}(f(B))$ is an IFGSPCS in $Y$, $f^{-1}(\mathrm{spcl}(f(B)))$ is an IFGSPCS in $X$. Since $X$ is an IFSPT$_{1/2}$ space, $f^{-1}(\mathrm{spcl}(f(B)))$ is an IFSPCS in $X$. Now, $f^{-1}(B) \subseteq f^{-1}(\mathrm{spcl}(B)) \subseteq \mathrm{spcl}(f^{-1}(\mathrm{spcl}(B)))$. We have $\mathrm{spcl}(f^{-1}(B)) \subseteq \mathrm{spcl}(f^{-1}(\mathrm{spcl}(B))) = f^{-1}(\mathrm{spcl}(B))$. This implies $\mathrm{spcl}(f^{-1}(B))$ $f^{-1}(\mathrm{spcl}(B))$ — (*). Again since $f$ is an IFMGSPHM, $f^{-1}$ is IFGSP irresolute mapping. Since $\mathrm{spcl}(f^{-1}(B))$ is an IFGSPCS in $X$, $(f^{-1})^{-1}(\mathrm{spcl}(f^{-1}(B))) = f(\mathrm{spcl}(f^{-1}(B)))$, is an IFGSPCS in $Y$. Now $B \subseteq (f^{-1})^{-1}(f^{-1}(B)) \subseteq (f^{-1})^{-1}(\mathrm{spcl}(f^{-1}(B))) = f(\mathrm{spcl}(f^{-1}(B)))$. Therefore $\mathrm{spcl}(B) \subseteq \mathrm{spcl}(f(\mathrm{spcl}(f^{-1}(B))))$ $= f(\mathrm{spcl}(f^{-1}(B))$, since $Y$ is an IFSPT$_{1/2}$ space. Hence $f^{-1}(\mathrm{spcl}(B)) \subseteq f^{-1}(f(\mathrm{spcl}(f^{-1}(B)) \subseteq \mathrm{spcl}(f^{-1}(B))$. That is $f^{-1}(\mathrm{spcl}(B)) \subseteq \mathrm{spcl}(f^{-1}(B))$ — (**). Thus from (*) and (**) we get $\mathrm{spcl}(f^{-1}(B)) = f^{-1}(\mathrm{spcl}(B))$ and hence the proof.

**Corollary 3.15:** If $f : X \to Y$ is an IFMGSPHM, where $X$ and $Y$ are IFSPT$_{1/2}$ spaces, then $\mathrm{spcl}(f(B)) = f(\mathrm{spcl}(B))$ for every IFS $B$ in $X$.

**Proof:** Since $f$ is an IFMGSPHM, $f^{-1}$ is also an IFMGSPHM. Therefore by Theorem 3.14 $\mathrm{spcl}((f^{-1})^{-1}(B)) = (f^{-1})^{-1}(\mathrm{spcl}(B))$ for every $B \subseteq X$. That is $\mathrm{spcl}(f(B)) = f(\mathrm{spcl}(B))$ for every IFS $B$ in $X$.

**Corollary 3.16:** If $f : X \to Y$ is an IFMGSPHM, where $X$ and $Y$ are IFSPT$_{1/2}$ spaces, then $\mathrm{spint}(f(B)) = f(\mathrm{spint}(B))$ for every IFS $B$ in $X$.

**Proof:** For any IFS $B \subseteq X$, $\mathrm{spint}(B) = (\mathrm{spcl}(B^c))^c$. By Corollary 3.15, $f(\mathrm{spint}(B)) = f(\mathrm{spcl}(B^c))^c$ $= (f(\mathrm{spcl}(B^c)))^c = (\mathrm{spcl}(f(B^c)))^c = \mathrm{spint}(f(B^c))^c = \mathrm{spint}(f(B^c)^c) = \mathrm{spint}(f(B))$.

**Corollary 3.17:** If $f : X \to Y$ is an IFMGSPHM, where $X$ and $Y$ are IFSPT$_{1/2}$ spaces, then $\mathrm{spint}(f^{-1}(B)) = f^{-1}(\mathrm{spint}(B))$ for every IFS $B$ in $Y$

**Proof:** Since $f$ is an IFMGSPHM, $f^{-1}$ is also an IFMGSPHM. Therefore the proof directly follows from Corollary 3.16.

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# SEVERAL OTHER FORMS OF SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

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**ABSTRACT** : By using the results in [23] and [26], we obtain the unified properties of the following ten families: the families of (1, 2)*-semi-open sets, (1, 2)*-preopen sets, (1, 2)*-$\alpha$-open sets. (1, 2)*-semi-preopen in [35]; the families of, (1, 2)-semi-open sets, (1, 2)-preopen sets, (1, 2)-$\alpha$-open sets, (1, 2)-semi-preopen sets in [33] and the new families of (1, 2)*-b-open sets, (1, 2)-$b$-open sets

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## 1. INTRODUCTION

In 1982, Tong [36] introduced the notion of $D$-sets and used these sets to introduce a separation axioms $D_1$ which is strictly between $T_0$ and $T_1$. In 1975, Maheshwari and Prasad [16] introduced new separation axioms semi-$T_0$, semi-$T_1$ and semi-$T_2$ by using semi-open sets due to Levine [15]. Borsan [4] and Caldas [5] introduced the notions of $s$-$D$-sets and a separation axiom which is strictly between semi-$T_0$ and semi-$T_1$. In 1990, Kar and Bhattacharyya [14] introduced new separation axioms pre-$T_0$, pre-$T_1$ and pre-$T_2$ by using preopen sets due to Mashhour et al. [20]. Recently, Caldas [6] and Jafari [12] introduced independently the notions of $p$-$D$-sets and separation axioms $p$-$D_1$ which is strictly between pre-$T_0$ and pre-$T_1$. In [28] and [17], the authors extended the notions of semi-$T_0$ and semi-$T_1$ topological spaces to bitopological spaces.

The notions of quasi-open sets [10], [32] or $\tau_1\tau_2$-open sets in bitopological spaces are introduced. The notions of (1, 2)*-preopen sets is introduced in [35]. In [27], the notions of (1, 2)*-pre-$T_k$ spaces ($k$ = 0, 1, 2), pre-diference sets [27] and pre-diference axioms are introduced and studied.

In [29] and [30], the present authors introduced the notions of minimal structures, $m$-spaces, $m$-continuous functions and $M$-continuous functions. In [23], the authors introduced

and studied the notions of $m$-$T_i$ spaces and $m$-$D_i$ spaces ($i = 0, 1, 2$) generalizing the notions of $T_i$, $p$-$T_i$, $D_i$-spaces ($i = 0, 1, 2$). The authors of [24], [25], [8] extended the notions of $m$-continuity and $M$-continuity in [29] and [30] to the notions of continuity forms in bitopological spaces.

In [26], the present authors extended the notions of $m$-$T_i$ spaces and $m$-$D_i$ spaces ($i = 0, 1, 2$) to the notions of some separation axioms in bitopological spaces. In the present paper, by using the results in [23] and [26], we obtain the unified properties of the following ten families: the families of (1, 2)*-semi-open sets, (1, 2)*-preopen sets, (1, 2)*-$\alpha$-open sets, (1, 2)*-semi-preopen in [35]; the families of, (1, 2)-semi-open sets, (1, 2)-preopen sets, (1, 2)-$\alpha$-open sets, (1, 2)-semi-preopen sets in [33] and the new families of (1, 2)*-$b$-open sets, (1, 2)-$b$-open sets.

## 2. PRELIMINARIES

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

**Definition 2.1** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be *$\alpha$-open* [22] (resp. *semi-open* [15], *preopen* [20], *$\beta$-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

The family of all semi-open (resp. preopen, $\alpha$-open, $\beta$-open, semi-preopen) sets in $X$ is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$).

**Definition 2.2** The complement of a semi-open (resp. preopen, $\alpha$-open, $\beta$-open, semi-preopen) set is said to be *semi-closed* [9] (resp. *preclosed* [11], *$\alpha$-closed* [21], *$\beta$-closed* [1], *semi-preclosed* [3]).

**Definition 2.3** The intersection of all semi-closed (resp. preclosed, $\alpha$-closed, $\beta$-closed, semi-preclosed) sets of $X$ containing $A$ is called the *semi-closure* [9] (resp. *preclosure* [11], *$\alpha$-closure* [21], *$\beta$-closure* [2], *semi-preclosure* [3]) of $A$ and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$).

**Definition 2.4** The union of all semi-open (resp. preopen, $\alpha$-open, $\beta$-open, semi-preopen) sets of $X$ contained in $A$ is called the *semi-interior* (resp. *preinterior*, *$\alpha$-interior*, *$\beta$-interior*, *semi-preinterior*) of $A$ and is denoted by sInt($A$) (resp. pInt($A$), $\alpha$Int($A$), $_\beta$Int($A$), spInt($A$)).

**Definition 2.5** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is called a *D-set* [36] (resp *s-D-set* [4], [5], *p-D-set* [12]) if there exist two open (resp. semi-open, pre-open) sets $U$, $V$ in $X$ such that $U \neq X$ and $A = U - V$.

If we replace open sets in the usual definitions of $T_0$, $T_1$, $T_2$ with $D$-sets (resp $s$-$D$-sets, $p$-$D$-sets), then we obtain the definitions of separation axioms $D_i$[36] (resp. $s$-$D_i$[4], [5], $p$-$D_i$ [14]) for $i = 0, 1, 2$.

Throughout the present paper $(X, \tau)$ and $(Y, \sigma)$ always denote topological spaces and $(X, \tau_1, \tau_2)$ and $(Y, \tau_1, \tau_2)$ denote bitopological spaces.

## 3. MINIMAL STRUCTURES

**Definition 3.1** A subfamily $m_X$ of the power set $\mathcal{P}(X)$ of a nonempty set $X$ is called a *minimal structure* (or briefly *m-structure*) [29] on $X$ if $\emptyset \in m_X$ and $X \in m_X$.

By $(X, m_X)$ (or briefly $(X, m)$), we denote a nonempty set $X$ with a minimal structure $m_X$ on $X$ and call it an *m-space.* Each member of $m_X$ is said to be *$m_X$-closed* (or briefly *m-closed*).

**Remark 3.1** Let $(X, \tau)$ be a topological space. Then the families $\tau$, SO($X$), PO($X$), $\alpha(X)$, $\beta(X)$ and SPO($X$) are all $m$-structures on $X$.

**Definition 3.2** Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the *$m_X$-closure* of $A$ and the *$m_X$-interior* of $A$ are defined in [19] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

**Remark 3.2** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_X = \tau$ (resp. SO($X$), PO($X$), $\alpha(X)$, $\beta(X)$, SPO($X$)), then we have

(1) mCl($A$) = Cl($A$) (resp. sCl($A$), pCl($A$), $_\alpha$Cl($A$), $_\beta$Cl($A$), spCl($A$)),

(2) mInt($A$) = Int($A$) (resp. sInt($A$), pInt($A$), $\alpha$Int($A$), $_\beta$Int($A$), spInt($A$)).

**Lemma 3.1** (Maki et al. [19]). *Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) *If* $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

**Lemma 3.2** (Popa and Noiri [29]). *Let $(X, m_X)$ be an m-space and $A$ a subset of $X$. Then* $x \in \mathrm{mCl}(A)$ *if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing $x$.*

**Definition 3.3** A minimal structure $m_X$ on a nonempty set $X$ is said to have *property* $\mathcal{B}$ [19] if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

**Lemma 3.3** (Popa and Noiri [31]). *Let $(X, m_X)$ be an m-space and $m_X$ have property $\mathcal{B}$. Then for a subset $A$ of $X$, the following properties hold:*

(1) *$A \in m_X$ if and only if* $\mathrm{mInt}(A) = A$,

(2) *$A$ is $m_X$-closed if and only if* $\mathrm{mCl}(A) = A$,

(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is $m_X$-closed.*

**Definition 3.4** A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* [29] if for each $x \in X$ and each $m_Y$-open sets $V$ of $Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$.

**Theorem 3.1** (Popa and Noiri [29]). *Let $(X, m_X)$ be an m-space and $m_X$ have property $\mathcal{B}$. For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *$f$ is M-continuous;*

(2) *$f^{-1}(V)$ is $m_X$-open for every $m_Y$-open set $V$ of $Y$;*

(3) *$f^{-1}(F)$ is $m_X$-closed for every $m_Y$-closed set $F$ of $Y$.*

## 4. MINIMAL STRUCTURES AND BITOPOLOGICAL SPACES

**Definition 4.1** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be *quasi-open* [10], [18] or $\tau_1\tau_2$-*open* (simply $\tau_{12}$-open) [33] if $A = B \cup C$, where $B \in \tau_1$ and $C \in \tau_2$.

The family of all $\tau_1\tau_2$-open sets of $(X, \tau_1, \tau_2)$ is denoted by $\tau_1\tau_2\mathrm{O}(X)$ (simply $\tau_{12}\mathrm{O}(X)$). It is obvious that $\tau_{12}\mathrm{O}(X)$ is an $m$-structure with property $\mathcal{B}$. The complement of a $\tau_1\tau_2$-open set of $(X, \tau_1, \tau_2)$ is said to be $\tau_1\tau_2$-closed (simply $\tau_{12}$-closed). The intersection of all $\tau_{12}$-closed sets containing a subset $A$ of $X$ is called the $\tau_1\tau_2$-closure of $A$ and is denoted by $\tau_1\tau_2\mathrm{Cl}(A)$ (simply $\tau_{12}\mathrm{Cl}(A)$). The union of all $\tau_1\tau_2$-open sets contained in $A$ is called the $\tau_1\tau_2$-interior of $A$ and is denoted by $\tau_1\tau_2\mathrm{Int}(A)$ (simply $\tau_{12}\mathrm{Int}(A)$).

**Definition 4.2** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be

(1) (1, 2)*-*semi-open* [35] if $A \subset \tau_{12}\mathrm{Cl}(\tau_{12}\mathrm{Int}(A))$,

(2) (1, 2)*-*preopen* [35] if $A \subset \tau_{12}\mathrm{Int}(\tau_{12}\mathrm{Cl}(A))$,

(3) (1, 2)*-$\alpha$-*open* [35] if $A \subset \tau_{12}\mathrm{Int}(\tau_{12}\mathrm{Cl}(\tau_{12}\mathrm{Int}(A)))$,

(4) (1, 2)*-*semi-preopen* [35] if $A \subset \tau_{12}\mathrm{Cl}(\tau_{12}\mathrm{Int}(\tau_{12}\mathrm{Cl}(A)))$,

(5) (1, 2)-*semi-open* [33] if $A \subset \tau_{12}\mathrm{Cl}(\tau_1\mathrm{Int}(A))$,

(6) (1, 2)-*preopen* [33] if $A \subset \tau_1\mathrm{Int}(\tau_{12}\mathrm{Cl}(A))$,

(7) (1, 2)-$\alpha$-*open* [33] if $A \subset \tau_1\mathrm{Int}(\tau_{12}\mathrm{Cl}(\tau_1\mathrm{Int}(A)))$,

(8) (1; 2)-*semi-preopen* [33] if $A \subset \tau_{12}\mathrm{Cl}(\tau_1\mathrm{Int}(\tau_{12}\mathrm{Cl}(A)))$.

The family of all (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-$\alpha$-open, (1, 2)*-semi-preopen, (1, 2)-semi-open, (1, 2)-preopen, (1, 2)-$\alpha$-open, (1, 2)-semi-preopen) sets is denoted by $(1, 2)^*\mathrm{SO}(X)$ (resp. $(1, 2)^*\mathrm{PO}(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*\mathrm{SPO}(X)$, $(1, 2)\mathrm{SO}(X)$, $(1, 2)\mathrm{PO}(X)$, $(1, 2)\alpha(X)$, $(1, 2)\mathrm{SPO}(X)$).

**Remark 4.1** Let $(X, \tau_1, \tau_2)$ be a bitopological space.

(1) The families $(1, 2)^*\mathrm{SO}(X)$, $(1, 2)^*\mathrm{PO}(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*\mathrm{SPO}(X)$, $(1, 2)\mathrm{SO}(X)$, $(1, 2)\mathrm{PO}(X)$, $(1, 2)\alpha(X)$, and $(1, 2)\mathrm{SPO}(X)$ are all $m$-structures with property $\mathcal{B}$.

(2) By $m(\tau_1, \tau_2)$ (simply $m_{12}$), we denote each member of the above eight familes and call it an $m$-structure determindcd by $\tau_1$ and $\tau_2$. Let $m(\tau_1, \tau_2) = \tau_{12}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$, $(1, 2)SO(X)$, $(1, 2)PO(X)$, $(1, 2)\alpha(X)$, $(1, 2)SPO(X)$), then we have

$m_{12}Cl(A) = \tau_{12}Cl(A)$ (resp. $(1, 2)^*sCl(A)$, $(1, 2)^*pCl(A)$, $(1, 2)^*\alpha Cl(A)$, $(1, 2)^*spCl(A)$, $(1, 2)sCl(A)$, $(1, 2)pCl(A)$, $(1, 2)\alpha Cl(A)$, $(1, 2)spCl(A)$),

$m_{12}Int(A) = \tau_{12}Int(A)$ (resp. $(1, 2)^*sInt(A)$, $(1, 2)^*pInt(A)$, $(1, 2)^*\alpha Int(A)$, $(1, 2)^*spInt(A)$, $(1, 2)sInt(A)$, $(1, 2)pInt(A)$, $(1, 2)\alpha Int(A)$, $(1, 2)spInt(A)$).

(3) Since $m(\tau_1, \tau_2)$ has property $\mathcal{B}$, by Lemma 3.3 we have

(i) $A$ is $m_{12}$-closed if and only if $m_{12}Cl(A) = A$,

(ii) $A$ is $m_{12}$-open if and only if $m_{12}Int(A) = A$

for $m(\tau_1, \tau_2) = \tau_{12}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$, $(1, 2)SO(X)$, $(1, 2)PO(X)$, $(1, 2)\alpha(X)$, $(1, 2)SPO(X)$).

(4) By Lemma 3.2, we obtain the result established in Proposition 2.2(ii) of [32].

(5) By Lemma 3.1, we obtain the relations between $m_{12}Cl(A)$ and $m_{12}Int(A)$.

## 5. $m(\tau_1, \tau_2)$-$T_i$-SPACES

**Definition 5.1** An $m$-space $(X, m_X)$ is said to be

(1) $m$-$T_0$ [23] if for any pair of distinct points $x$, $y$ of $X$, there exists an $m_X$-open set containing $x$ but not $y$ or an $m_X$-open set containing $y$ but not $x$,

(2) $m$-$T_1$ [23] if for any pair of distinct points $x$, $y$ of $X$, there exists an $m_X$-open set containing $x$ but not $y$ and an $m_X$-open set containing $y$ but not $x$,

(3) $m$-$T_2$ [29] if for any pair of distinct points $x$, $y$ of $X$, there exist $m_X$-open sets $U$, $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Definition 5.2** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then $(X, \tau_1, \tau_2)$ is said to be $m(\tau_1, \tau_2)$-$T_i$ (briefly $m_{12}$-$T_i$) if the $m$-space $(X, m(\tau_1, \tau_2))$ is $m$-$T_i$ for $i = 0, 1, 2$.

**Remark 5.1** Let $(X, \tau_1, \tau_2)$ be a bitopological space. Let $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{SPO}(X)$ (resp. $(1, 2)\mathrm{SO}(X)$, $(1, 2)\mathrm{PO}(X)$, $(1, 2)\alpha(X)$).

(1) If $(X, \tau_1, \tau_2)$ is $m_{12}$-$T_0$, then $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-pre-$T_0$ [27] (resp. ultra semi-$T_0$ [13], ultra pre-$T_0$ [34], ultra $\alpha$-$T_0$ [13]).

(2) If $(X, \tau_1, \tau_2)$ is $m_{12}$-$T_1$, then $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-pre-$\mathrm{T}_1$ [27] (resp. ultra semi-$T_1$ [13], ultra pre-$T_1$ [34], ultra $\alpha$-$T_1$ [13]).

(3) If $(X, \tau_1, \tau_2)$ is $m_{12}$-$T_2$, then $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-pre-$T_2$ [27] (resp. ultra semi-$T_2$ [13], ultra pre-$T_2$ [34], ultra $\alpha$-$T_2$ [13]).

We shall recall the definition of $\Lambda_m$-sets, a topological space $(X, \Lambda_m)$ and $(\Lambda, m)$-closed sets in order to obtain characterizations of $m_{12}$-$T_i$ spaces for $i = 0, 1, 2$. Let $(X, m)$ be an $m$-space and $A$ a subset of $X$. A subset $\Lambda_m(A)$ is defined in [7] as follows: $\Lambda_m(A) = \cap\{U : A \subset U \in m\}$. The subset $A$ is called a *$\Lambda_m$-set* [7] if $A = \Lambda_m(A)$. The family of all $\Lambda_m$-sets of $(X, m_X)$ is denoted by $\Lambda_m(X)$ (or simply $\Lambda_m$). It follows from Theorem 3.1 of [7] that the pair $(X, \Lambda_m)$ is an Alexandorff (topological) space. The subset $A$ is said to be *$(\Lambda, m)$-closed* [7] if $A = U \cap F$, where $U$ is a $\Lambda_m$-set and $F$ is an $m$-closed set of $(X, m)$.

Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. For an $m$-structure $m(\tau_1, \tau_2)$, $\Lambda_{m(\tau_1,\tau_2)}$-sets, a topological space $\left(X, \Lambda_{m(\tau_1,\tau_2)}\right)$ and $(\Lambda, m(\tau_1, \tau_2))$-closed sets are similarly defined.

**Lemma 5.1** (Noiri and Popa [23]). *An m-space $(X, m_X)$ is $m$-$T_0$ if and only if* $\mathrm{mCl}(\{x\}) \neq \mathrm{Cl}(\{y\})$ *for any pair of distinct points $x, y \in X$.*

**Lemma 5.2** (Cammaroto and Noiri [7]). *For an m-space $(X, m_X)$, the following properties are equivalent:*

(1) *$(X, m_X)$ is $m$-$T_0$;*

(2) *The singleton $\{x\}$ is $(\Lambda, m)$-closed for each $x \in X$;*

(3) *$(X, \Lambda_m)$ is $T_0$.*

**Theorem 5.1** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$* and *$\tau_2$. Then the following properties are equivalent:*

(1) $(X, \tau_1, \tau_2)$ *is* $m_{12}$-$T_0$;

(2) $m_{12}\mathrm{Cl}(\{x\}) \neq m_{12}\mathrm{Cl}(\{y\})$ *for any pair of distinct points* $x, y \in X$;

(3) *The singleton* $\{x\}$ *is* $(\Lambda, m(\tau_1, \tau_2))$*-closed for each* $x \in X$;

(4) $\left(X, \Lambda_{m(\tau_1,\tau_2)}\right)$ *is* $T_0$.

**Proof.** This is an immediate consequence of Definition 5.2 and Lemmas 5.1 and 5.2.

**Corollary 5.1** A bitopological space $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-pre-$T_0$ [27] (resp. ultra $\alpha T_0$ [34]?) if distinct points have distinct $(1, 2)^*$-preclosure (resp. $(1, 2)$-$\alpha$-closure).

**Lemma 5.3** (Noiri and Popa [23]). *Let* $(X, m_X)$ *be an m-space and* $m_X$ *have property* $\mathcal{B}$. *Then* $(X, m_X)$ *is* $m$-$T_1$ *if and only if for each points* $x \in X$, *the singleton* $\{x\}$ *is* $m_X$*-closed.*

**Lemma 5.4** (Cammaroto and Noiri [7]). *Let* $(X, m_X)$ *be an m-space and* $m_X$ *have property* $\mathcal{B}$. *Then for the m-space* $(X, m_X)$, *the following properties are equivalent.*

(1) $(X, m_X)$ *is* $m$-$T_1$;

(2) *The singleton* $\{x\}$ *is a* $\Lambda_m$*-set for each* $x \in X$;

(3) $(X, \Lambda_m)$ *is discrete.*

**Theorem 5.2** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *a minimal structure on* $X$ *determined by* $\tau_1$ *and* $\tau_2$. *Then for the space* $(X, \tau_1, \tau_2)$, *the following properties are equivalent:*

(1) $(X, \tau_1, \tau_2)$ *is* $m_{12}$-$T_1$;

(2) *The singleton* $\{x\}$ *is* $m_{12}$*-closed for each points* $x \in X$;

(3) *The singleton* $\{x\}$ *is a* $\Lambda_{m(\tau_1,\tau_2)}$*-set for each* $x \in X$;

(4) $\left(X, \Lambda_{m(\tau_1,\tau_2)}\right)$ *is discrete.*

**Proof.** This is an immediate consequence of Lemmas 5.3 and 5.4.

**Remark 5.2** Let $(X, \tau_1, \tau_2)$ be a bitopological space. Let $m(\tau_1, \tau_2) = \tau_{12}\mathrm{O}(X)$ (resp. $(1, 2)^*\mathrm{PO}(X)$, $(1, 2)\alpha(X)$, $(1, 2)\mathrm{PO}(X)$), then by Theorem 5.2, we obtain the results established in [18] (resp. Theorem 3.11 of [27], Theorem 3.8 of [13] or Theorem 4.8 of [33], Theorem 6.8 of [33]).

**Lemma 5.5** (Noiri and Popa [26]). *Let $(X, m_X)$ be an m-space and $m_X$ have property B Then, for the m-space $(X, m_X)$ the following properties are equivalent:*

(1) *$(X, m_X)$ is $m$-$T_2$;*

(2) *For any distinct points $x, y \in X$, there exists $U \in m_X$ containing $x$ such that $y \notin \mathrm{mCl}(U)$;*

(3) *For each point $x \in X$, $\{x\} = \cap\{\mathrm{mCl}(U) : x \in U \in m_X\}$;*

(4) *For each pair of distinct points $x, y \in X$, there exists an M-continuous function $f$ of $(X, m_X)$ into an $m$-$T_2$ m-space $(Y, m_Y)$ such that $f(x) \neq f(y)$.*

**Definition 5.3** Let $(X, \tau_1, \tau_2)$ (resp. $(Y, \sigma_1, \sigma_2)$) be a bitopological space and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) a minimal structure on $X$ (resp. on $Y$) determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *$M_{12}$-continuous* if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ is $M$-continuous.

**Theorem 5.3** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then, for the space $(X, \tau_1, \tau_2)$ the following properties are equivalent:*

(1) *$(X, \tau_1, \tau_2)$ is $m_{12}$-$T_2$;*

(2) *For any distinct points $x, y \in X$, there exists $U \in m(\tau_1, \tau_2)$ containing $x$ such that $y \notin m_{12}\mathrm{Cl}(U)$;*

(3) *For each point $x \in X$, $\{x\} = \cap\{m_{12}\mathrm{Cl}(U) : x \in U \in m(\tau_1, \tau_2)\}$;*

(4) *For each pair of distinct points $x, y \in X$, there exists an $M_{12}$-continuous function $f$ of $(X, \tau_1, \tau_2)$ into an $m_{12}$-$T_2$ space $(Y, \sigma_1, \sigma_2)$ such that $f(x) \neq f(y)$.*

**Proof.** This is an immediate consequence of Lemma 5.5.

**Remark 5.3** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$ (resp. $(1, 2)\alpha(X)$). Then, by Theorem 5.3, we obtain the result established in Theorem 3.15 of [27] (resp. Theorem 6.10 of [33]).

**Lemma 5.6** (Noiri and Popa [26]). *Let $f : (X, m_X) \to (Y, m_Y)$ be an injective M-continuous function and $m_X$ have property B If $(Y, m_Y)$ is $m$-$T_i$, then $(X, m_X)$ is $m$-$T_i$ for $i = 0, 1, 2$.*

**Theorem 5.4** *Let $m(\tau_1, \tau_2)$ and $(\sigma_1, \sigma_2)$ be minimal structures on $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$, respectively. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an $M_{12}$-continuous injection and $(Y, \sigma_1, \sigma_2)$ is $m_{12}$-$T_i$, then $(X, \tau_1, \tau_2)$ is $m_{12}$-$T_i$ for $i = 0, 1, 2$.*

**Proof.** This follows immediately from Definition 5.2 and Lemma 5.6.

**Definition 5.4** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(1, 2)^*$-*preirresolute* (resp. $(1, 2)$-*preirresolute* if the inverse image of every $(1, 2)^*$-preopen (resp. $(1, 2)$-preopen) set in $(Y, \sigma_1, \sigma_2)$ is $(1, 2)^*$-preopen (resp. $(1, 2)$-preopen) in $(X, \tau_1, \tau_2)$.

**Corollary 5.2** *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is injective and $(1, 2)^*$-preirresolute (resp. $(1, 2)$-preirresolute) and $(Y, \sigma_1, \sigma_2)$ is $(1, 2)^*$-pre-$T_2$ (resp. ultra pre-$T_2$), then $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-pre-$T_2$ (resp. ultra pre-$T_2$).*

**Proof.** This is shown in Theorem 3.18 of [27] (resp. theorem 6.15 of [33]).

**Definition 5.5** An $m$-space $(X, m_X)$ is said to be *m-regular* [31] if for each $m_X$-closed set $F$ and for each point $x \notin F$, there exist disjoint $m_X$-open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

**Theorem 5.5** *If an m-space $(X, m_X)$ is $m$-$T_0$ and m-regular, then it is $m$-$T_2$.*

**Proof.** Let $x$ and $y$ be any pair of distinct points of $X$, then there exists $U \in m_X$ containing $x$ but not $y$. Then $X - U$ is $m_X$-closed and $x \notin X - U$. Since $X$ is $m$-regular, there exist disjoint $m_X$-open sets $V_1$ and $V_2$ such that $x \in V_1$ and $X - U \subset V_2$. Thus $x \in V_1$, $y \in V_2$ and $V_1 \cap V_2 = \emptyset$. Hence $(X, m_X)$ is $m$-$T_2$.

**Theorem 5.6** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. If $(X, \tau_1, \tau_2)$ is $m(\tau_1, \tau_2)$-regular, then the following properties are equivalent:*

(1) *$(X, \tau_1, \tau_2)$ is $m_{12}$-$T_0$;*

(2) *$(X , \tau_1, \tau_2)$ is $m_{12}$-$T_1$;*

(3) *$(X, \tau_1, \tau_2)$ is $m_{12}$-$T_2$.*

**Proof.** It is shown in [23] that $m\text{-}T_2 \Rightarrow m\text{-}T_1 \Rightarrow m\text{-}T_0$. Therefore, the proof follows from Definition 5.2 and Theorem 5.5.

**Corollary 5.3** (Paŋmani and Thivagar [27]). *Every* $(1, 2)^*$*-pre-*$T_0$ *bitopological space is* $(1, 2)^*$*-pre-*$T_2$ *if it is* $(1, 2)^*$*-pre-regular.*

**Definition 5.6** An $m$-space $(X, m_X)$ is said to be *m-symmetric* if for each point $x, y \in X$, $x \in m\mathrm{Cl}(\{y\})$ implies $y \in m\mathrm{Cl}(\{x\})$.

**Lemma 5.7** (Noiri and Popa [23]). *Let* $(X, m_X)$ *be an m-space, where* $m_X$ *has property* $\mathcal{B}$. *Then the following properties are equivalent:*

(1) $(X, m_X)$ *is m-symmetric and* $m\text{-}T_0$;

(2) $(X, m_X)$ *is* $m\text{-}T_1$

**Theorem 5.7** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *a minimal structure on X determined by* $\tau_1$ *and* $\tau_2$. *Then the following properties are equivalent:*

(1) $(X, \tau_1, \tau_2)$ *is* $m(\tau_1, \tau_2)$*-symmetric and* $m_{12}\text{-}T_0$;

(2) $(X, \tau_1, \tau_2)$ *is* $m_{12}\text{-}T_1$.

**Proof.** This follows from Definition 5.2 and Lemma 5.7.

## 6. $m(\tau_1, \tau_2)$-diference axioms

**Definition 6.1** A subset $A$ of an $m$-space $(X, m_X)$ is called an *m-D-set* [23] if there exist two $m_X$-open sets $U$ and $V$ such that $U \neq X$ and $A = U - V$.

Every $m_X$-open set different from $X$ is an $m$-$D$-set since we can take as follows $A = U$ and $V = \emptyset$.

**Definition 6.2** An $m$-space $(X, m_X)$ is said to be

(1) $m\text{-}D_0$ [23] if for any distinct points $x, y \in X$, there exists an $m$-$D$-set of $X$ containing $x$ but not $y$ or an $m$-$D$-set of $X$ containing $y$ but not $x$,

(2) $m\text{-}D_1$ [23] if for any distinct points $x, y \in X$, there exists an *m-D-set* of $X$ containing $x$ but not $y$ and an $m$-$D$-set of $X$ containing $y$ but not $x$,

(1) $m$-$D_2$ [23] if for any distinct points $x, y \in X$, there exist $m$-$D$-sets $U$, $V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Remark 6.1** Let $(X, \tau)$ be a topological space and $m_X$ is a minimal structure on $X$. If $m_X = \tau$ (resp. SO($X$), PO($X$) $\alpha(X)$), then we obtain we obtain the definitions of separation axioms $D_i$ [36] (resp. $s$-$D_i$ [4], $p$-$D_i$ [12], $\alpha$-$D_i$) for $i = 0, 1, 2$.

**Remark 6.2** By Definitions 5.1 and 6.2, we have the following diagram [23]:

$$\begin{array}{ccccc} m\text{-}T_2 & \Rightarrow & m\text{-}T_1 & \Rightarrow & m\text{-}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ m\text{-}D_2 & \Rightarrow & m\text{-}D_1 & \Rightarrow & m\text{-}D_0 \end{array}$$

**Definition 6.3** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then a subset $A$ of $X$ is called an *$m(\tau_1, \tau_2)$-$D$-set* (briefly $m_{12}$-$D$-set) in $(X, \tau_1, \tau_2)$ if $A$ is an $m$-$D$-set in the $m$-space $(X, m(\tau_1, \tau_2))$.

**Remark 6.3** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. If $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$ (resp. $(1, 2)\alpha(X)$). Then, by Definition 6.3, we obtain the definition of $(1, 2)^*$-pre-diference sets [27] (resp. ultra $\alpha D$-space [33]).

**Definition 6.4** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then $(X, \tau_1, \tau_2)$ is said to be $m(\tau_1, \tau_2)$-$D_i$ (briefly $m_{12}$-$D_i$) if the $m$-space $(X, m(\tau_1, \tau_2))$ is $m$-$D_i$ for $i = 0, 1, 2$.

If $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$, then we obtain the definitions of $(1, 2)^*$-pre-$D_i$ spaces for $i = 0, 1, 2$ in [27].

**Definition 6.5** A bitoppological space $(X, \tau_1, \tau_2)$ is said to be

(1) *$(1, 2)^*$-pre-$D_0$* [27] if for any distinct points $x, y \in X$, there exists an $(1, 2)^*$-$pD$-set of $X$ containing one of $x$ and $y$ but not the other,

(2) *$(1, 2)^*$-pre-$D_1$* [27] if for any distinct points, $x, y \in X$, there exist $(1, 2)^*$-$pD$-sets $U$, $V$ of $X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$,

(3) *$(1, 2)^*$-pre-$D_2$* [27] if for any distinct points $x, y \in X$, there exist $(1, 2)^*$-$pD$-sets $U$, $V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Remark 6.4** (1) By Remark 6.2 and Definition 6.4, we have the following diagram:

$$\begin{array}{ccccc} m_{12}\text{-}T_2 & \Rightarrow & m_{12}\text{-}T_1 & \Rightarrow & m_{12}\text{-}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ m_{12}\text{-}D_2 & \Rightarrow & m_{12}\text{-}D_1 & \Rightarrow & m_{12}\text{-}D_0 \end{array}$$

(2) It follows from Example 3.7 of [27] that $m_{12}$-$D_i$ does not imply $m_{12}$-$T_i$ for $i = 0$, 1, 2.

(3) It follows from Example 4.6 of [27] that $m_{12}$-$D_{i-1}$ does not imply $m_{12}$-$D_i$ for $i =$ 1, 2.

(4) It follows from Example 3.10 of [27] that $m_{12}$-$D_{i-1}$ does not imply $m_{12}$-$T_i$ for $i$ = 1, 2.

**Lemma 6.1** (Noiri and Popa [23]). *An m-space* $(X, m_X)$ *m*-$D_0$ *if and only if it is m*-$T_0$.

**Theorem 6.1** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *a minimal structure on X determined by* $\tau_1$ *and* $\tau_2$. *Then* $(X, \tau_1, \tau_2)$ *is* $m_{12}$-$D_0$ *if and only if* $m_{12}$-$T_0$.

**Proof.** This follows from Definition 5.2 and Lemma 6.1.

**Remark 6.5** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$ (resp. (1, 2)$\alpha(X)$). Then, by Theorem 6.1, we obtain the result established in Theorem 4.1 of [27] (resp. Theorem 5.4 of [33]).

**Lemma 6.2** (Noiri and Popa [23]). *Let* $(X, m_X)$ *be an m-space and* $m_X$ *have property* B. *Then* $(X, m_X)$ *m*-$D_1$ *if and only if it is m*-$D_2$.

**Theorem 6.2** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *a minimal structure on X determined by* $\tau_1$ *and* $\tau_2$. *Then* $(X, \tau_1, \tau_2)$ *is* $m_{12}$-$D_1$ *if and only if it is* $m_{12}$-$D_2$.

**Proof.** This follows from Definition 6.4 and Lemma 6.2.

**Remark 6.6** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$ (resp. (1, 2)$\alpha(X)$). Then, by Theorem 6.2, we obtain the result established in Theorem 4.8 of [27] (resp. theorem 5.10 of [33]).

**Lemma 6.3** (Noiri and Popa [23]). *Let $(X, m_X)$ be an m-symmetric m-space and $m_X$ have property* B. *Then $m$-$T_0$, $m$-$T_1$, $m$-$D_0$, $m$-$D_1$, and $m$-$D_2$ are all equivalent.*

**Theorem 6.3** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then for $(X, \tau_1, \tau_2)$, the following axioms are equivalent·*

**Proof.** This follows from Definition 6.4 and Lemma 6.3.

**Definition 6.6** Let $(X, m_X)$ be an $m$-space. A point $x \in X$ is called an *mcc-point* if $X$ is the unique $m_X$-open set containing $x$.

**Remark 6.7** If $(X, \tau)$ is a topological space and $m_X = \tau$ (resp. SO($X$), PO($X$)). Then, an *mcc*-point is called a *cc*-point [36] (resp. *s.cc*-point [4] or *sc.c*-point [5], *pcc*-point [6]).

**Definition 6.7** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. A point $x \in X$ is called an $m(\tau_1, \tau_2)$ *cc-point* (simply $m_{12}$*cc-point*) if $X$ is the unique $m(\tau_1, \tau_2)$-open set containing $x$.

**Lemma 6.4** (Noiri and Popa [23]). *If an m-space $(X, m_X)$ is $m$-$T_0$, then there exists at most one mcc-point.*

**Theorem 6.4** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then $(X, \tau_1, \tau_2)$ is $m_{12}$-$T_0$, then there exists at most one $m_{12}$cc-point.*

**Proof.** This follows from Definition 6.7 and Lemma 6.4.

**Corollary 6.1** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$. Then if $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-pre-$T_0$, then there exists at most one $(1, 2)^*$pcc-point*

**Lemma 6.5** (Noiri and Popa [23]). *An $m$-$T_0$ m-space $(X, m_X)$ is $m$-$D_1$ if and only if it does not have any mcc-point.*

**Theorem 6.5** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then $(X, \tau_1, \tau_2)$ is $m_{12}$-$D_1$ if and only if it does not have any $m_{12}$cc-point.*

**Proof.** This follows from Definition 6.6 and Lemma 6.5.

**Corollary 6.2** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$. *Then* $(X, \tau_1, \tau_2)$ *is* $(1, 2)^*$*-pre-*$D_1$ *if and only if it does not have any* $(1, 2)^*$*pcc-point.*

**Lemma 6.6** (Noiri and Popa [23]). *Let* $(X, m_X)$ *and* $(Y, m_Y)$ *be m-spaces, where* $m_X$ *has property* $\mathcal{B}$, *and* $f : (X, m_X) \to (Y, m_Y)$ *an M-continuous surjection. If* $B$ *is an m-D-set of* $(Y, m_Y)$, *then* $f^{-1}(B)$ *is an m-D-set of* $(X, m_X)$.

**Theorem 6.6** *Let* $(X, \tau_1, \tau_2)$ *(resp.* $(Y, \sigma_1, \sigma_2)$*) be a bitopological space and* $m(\tau_1, \tau_2)$ *(resp.* $m(\sigma_1, \sigma_2)$*) an minimal structure on X (resp. Y) determined by* $\tau_1$ *and* $\tau_2$ *(resp.* $\sigma_1$ *and* $\sigma_2$*). If* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is an* $M_{12}$*-continuous surjection and* $(Y, \sigma_1, \sigma_2)$ *is* $m(\sigma_1, \sigma_2)$*-*$D_1$, *then* $(X, \tau_1, \tau_2)$ *is* $m(\tau_1, \tau_2)$*-*$D_1$.

**Proof.** This is an immediate consequence of Lemma 6.6.

In case $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{PO}(X)$ and $m(\sigma_1, \sigma_2) = (1, 2)^*\mathrm{PO}(Y)$, we have the following corollary.

**Corollary 6.3** *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a* $(1, 2)^*$*-preirresolute surjection If* $(Y, \sigma_1, \sigma_2)$ *is* $(1, 2)^*$*-pre-*$D_1$, *then* $(X, \tau_1, \tau_2)$ *is* $(1, 2)^*$*-pre-*$D_1$.

**Lemma 6.7** (Noiri and Popa [23]). *An m-space* $(X, m_X)$, *where* $m_X$ *has property* $\mathcal{B}$, *is m-*$D_1$ *if and only if for each pair of distinct point* $x, y \in X$, *there exists an M-continuous surjection of* $(X, m_X)$ *onto an m-*$D_1$ *m-space* $(Y, m_Y)$ *such that* $f(x) \neq f(y)$.,

**Theorem 6.7** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *an minimal structure on X determined by* $\tau_1$ *and* $\tau_2$. *Then* $(X, \tau_1, \tau_2)$ *is* $m_{12}$*-*$D_1$ *if and only if for each pair of distinct point* $x, y \in X$, *there exists an* $M_{12}$*-continuous surjection of* $(X, \tau_1, \tau_2)$ *onto an* $m_{12}$*-*$D_1$ *bitopological space* $(Y, \sigma_1, \sigma_2)$ *such that* $f(x) \neq f(y)$.

**Proof.** This is an immediate consequence of Lemma 6.7.

**Corollary 6.4** *A bitopological space* $(X, \tau_1, \tau_2)$ *is* $(1, 2)^*$*-pre-*$D_1$ *if and only if for each pair of distinct point* $x, y \in X$, *there exists a* $(1, 2)^*$*-preirresolute surjection of* $(X, \tau_1, \tau_2)$ *onto an* $(1, 2)^*$*-pre-*$D_1$ *space* $(Y, \sigma_1, \sigma_2)$ *such that* $f(x) \neq f(y)$.

## 7. NEW FORMS OF $m_{12}$-$T_i$ AND $m_{12}$-$D_i$ FOR $i = 0, 1, 2$.

**Definition 7.1** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be

(1) $(1, 2)^*$-*b-open* if $A \subset \tau_{12}\mathrm{Cl}(\tau_{12}\mathrm{Int}(A)) \cup \tau_{12}\mathrm{Int}(\tau_{12}\mathrm{Cl}(A))$,

(2) $(1, 2)$-*b-open* if $A \subset \tau_{12}\mathrm{Cl}(\tau_1\mathrm{Int}(A)) \cup \tau_1\mathrm{Int}(\tau_{12}\mathrm{Cl}(A))$.

The family of all $(1, 2)^*$-$b$-open (resp. $(1, 2)$-$b$-open) sets is denoted by $(1, 2)^*\mathrm{BO}(X)$ (resp. $(1, 2)\mathrm{BO}(X)$).

**Remark 7.1** Let $(X, \tau_1, \tau_2)$ be a bitopological space.

(1) The families $(1, 2)^*\mathrm{BO}(X)$ and $(1, 2)\mathrm{BO}(X)$ are $m$-structures with property B.

(2) By $m(\tau_1, \tau_2)$ (simply $m_{12}$), we denote each one of the above two familes and call it an $m$-structure determinded by $\tau_1$ and $\tau_2$. Let $m(\tau_1, \tau_2) = (1, 2)^*\mathrm{BO}(X)$ (resp. $(1, 2)\mathrm{BO}(X)$), then we have

$m_{12}\mathrm{Cl}(A) = (1, 2)^*\mathrm{bCl}(A)$ (resp. $(1, 2)\mathrm{bCl}(A)$),

$m_{12}\mathrm{Int}(A) = (1, 2)^*\mathrm{bInt}(A)$ (resp. $(1, 2)\mathrm{bInt}(A)$).

(3) $m_{12}$-$T_i = (1, 2)^*$-$b$-$T_i$ (resp. $(1, 2)$-$b$-$T_i$) for $i = 0, 1, 2$.

(4) $m_{12}$-$D_i = (1, 2)^*$-$b$-$D_i$ (resp. $(1, 2)$-$b$-$D_i$) for $i = 0, 1, 2$.

Now, we can apply the results established in Sections 5 and 6 to the above two families.

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# COMMON FIXED POINTS FOR GENERALIZED ($f$, $g$)-NONEXPANSIVE MAPPINGS

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**ABSTRACT :** The aim of this paper is to prove the unique common fixed point theorems for generalized ($f$, $g$)-contraction if both ($T$, $f$) and ($T$, $g$) are weakly compatible in a metric space ($E$, $d$). We also establish the result for generalized ($f$, $g$)-Nonexpansive mappings in a linear normed space $E$

**Key words :** Common fixed point; generalized ($f$, $g$)-contraction; generalized ($f$, $g$)-nonexpansive, weakly compatible mappings.

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## 1. INTRODUCTION & PRELIMINARIES

Let $K$ be a nonempty subset of a metric space ($E$, $d$) and $T$ a mapping from $K$ to $E$. We shall denote the closure of $K$ by $\overline{K}$, the boundary of $K$ by $\partial K$, and all positive integer by N, and the set of fixed points of $T$, $\{x \in K : x = Tx\}$, by $F(T)$. When $\{x_n\}$ is a sequence in $E$, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) will denote the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to $x$.

A mapping $T : K \to E$ is called an ($f$, $g$)-contraction if there exists $0 \leq k \leq 1$ such that $d(Tx, Ty) \leq kd(fx, gy)$ for all $x, y \in K$. If $k = 1$, then $T$ is called ($f$, $g$)-nonexpansive. If $g = f$, in the above inequality, $T$ is said to be an $f$-contraction (respectively, $f$-nonexpansive). A point $x \in K$ is a coincidence point (respectively, common fixed point) of $f$ and $T$ if $f(x) = Tx$ (respectively, $x = f(x) = Tx$) The set of coincidence points of $f$ and $T$ is denoted by $C(f, T)$. The pair ($f$, $T$) is called to be

(i) *Compatible* [3], if $fx_n, Tx_n \in K$ and $\lim_{n\to\infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in K$;

(ii) *Weakly compatible,* if $T(C(f, T)) \subset K$ and $f(C(f, T)) \subset K$ such that $fTx = Tfx$ whenever $x \in C(f, T)$. Suppose that $E$ is a compact metric space and both $T$ and

$f$ are continuous self-mapping, then $(f, T)$ compatible equivalent to $(f, T)$ weakly compatible [3, Theorem 2.2, Cor. 2.3].

Let $K$ be a nonempty closed convex subset of a normed space $E$. A mapping $f : K \to K$ is affine if $K$ is convex and $f(kx + (1-k)y) = kfx + (1-k)fy$ for all $x, y \in K$ and all $k \in [0, 1]$. A subset $K$ of a norm space $E$ is called $q$-starshaped with $q \in K$ if $kx + (1-k)q \in K$ for all $x \in K$ and all $k \in [0, 1]$. Let $T$ be a mapping from a $q$-starshaped subset $K$ of a normed space $E$ into itself. $T$ is called $q$-affine if $T(kx + (1-k)q) = kTx + (1-k)q$ for all $x \in K$ and all $k \in [0, 1]$. It is easy to see that if $T$ is $q$-affine, then $Tq = q$.

Let $K$ be a $q$-starshaped subset of a normed space $E$ and $T, f$ be two mappings from $K$ to itself. Then $(T, f)$ is called $C_q$-commuting [1] if $fTx = Tfx$ for all $x \in C_q(f, T)$, whenever $C_q(f, T) = \cup\{C(f, T_k) : 0 \leq k < 1\}$ and $T_k x = (1-k)q + kTx$. Clearly, $C_q$-commuting maps are weakly compatible but the converse does not hold in general.

The aim of this paper is to prove that there is a unique common fixed point of $T, f, g$ if $T$ is generalized $(f, g)$-contractive and both $(T, f)$ and $(T, g)$ are weakly compatible in a metric space $(E, d)$. We also prove the result for generalized $(f, g)$-nenexpansive.

## 2. MAIN RESULTS

Let $K$ be a nonempty subset of a metric space $(E, d)$ and $T, f, g$ be three mappings on $K$. In this section, we will study the common fixed point theorems of a generalized $(f, g)$-contraction and a generalized $(f, g)$-nonexpansive mapping. Now we define the generalized $(f, g)$-contraction.

A mapping $T : K \to E$ is called generalized $(f, g)$-contraction, if there exists constants $a, b, c \in (0, 1)$ such that $a + 2b + c < 1$ satisfying the condition,

$$d(Tx, Ty) \leq a \max\{d(fx, Tx), d(gy, Ty)\} + b \min\{d(fx, Ty), d(gy, Tx)\} + cd(fx, gy) \quad \text{...(2.1)}$$

for all $x, y \in K$

Next, we give our main results.

**Theorem 2.1.** *Let $K$ be a nonempty subset of a metric space $(E, d)$ and $T, f, g: K \to E$ be three mappings with $T(K) \subset f(K) \cap g(K)$. Suppose that $\overline{T(K)}$ is complete, and $T$ is generalized $(f, g)$-contraction with constants $a, b, c \in (0, 1)$ such that $a + 2b + c < 1$. Then neither $C(T, f)$ nor $C(T, g)$ is empty. Moreover, if, in addition, both $(T, f)$ and $(T, g)$ are weakly compatible then $F(T) \cap F(f) \cap F(g)$ is singleton.*

**Proof.** Choose $x_0 \in K$. Since $\overline{T(K)} \subset f(K) \cap g(K)$, there exists a sequence $\{x_n\}$ in $K$ such that $Tx_{2n} = fx_{2n+1}$ and $Tx_{2n+1} = gx_{2n+2}$ for all $n \geq 0$.

Now from (2.1),

$$\begin{aligned}
d(Tx_{2n+1}, Tx_{2n}) &\leq a \max\{d(fx_{2n+1}, Tx_{2n+1}), d(gx_{2n}, Tx_{2n})\} \\
&\quad + b \min\{d(fx_{2n+1}, Tx_{2n}), d(gx_{2n}, Tx_{2n+1})\} + cd(fx_{2n+1}, gx_{2n}) \\
&\leq a \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Tx_{2n})\} \\
&\quad + b \min\{d(Tx_{2n}, Tx_{2n}), d(Tx_{2n-1}, Tx_{2n+1})\} + cd(Tx_{2n}, Tx_{2n-1}) \\
&\leq a \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Tx_{2n})\} + bd(Tx_{2n-1}, Tx_{2n+1}) \\
&\quad + cd(Tx_{2n}, Tx_{2n-1}) \\
&\leq a \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Tx_{2n})\} \\
&\quad + b[d(Tx_{2n+1}, Tx_{2n}) + d(Tx_{2n}, Tx_{2n-1})] + cd(Tx_{2n}, Tx_{2n-1})
\end{aligned}$$

Now there are two cases.

Case-(I)

$$d(Tx_{2n+1}, Tx_{2n}) \leq (a + b)d(Tx_{2n+1}, Tx_{2n}) + (b + c)d(Tx_{2n}, Tx_{2n-1})$$

$$\Rightarrow d(Tx_{2n+1}, Tx_{2n}) \leq \frac{(b+c)}{[1-(a+b)]} d(Tx_{2n}, Tx_{2n-1})$$

Case-(II)

$$d(Tx_{2n+1}, Tx_{2n}) \leq (a + b + c)d(Tx_{2n}, Tx_{2n-1}) + bd(Tx_{2n+1}, Tx_{2n})$$

$$\Rightarrow d(Tx_{2n+1}, Tx_{2n}) \leq \frac{(a+b+c)}{(1-b)} d(Tx_{2n}, Tx_{2n-1})$$

Again,

$$\begin{aligned} d(Tx_{2n-1}, Tx_{2n}) &\le a \max\{d(fx_{2n-1}, Tx_{2n-1}), d(gx_{2n}, Tx_{2n})\} \\ &+ b \min\{d(fx_{2n-1}, Tx_{2n}), d(gx_{2n}, Tx_{2n-1})\} \; cd(fx_{2n-1}, gx_{2n}) \\ &\le a \max\{d(Tx_{2n-2}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n})\} \\ &+ b \min\{d(Tx_{2n-2}, Tx_{2n}), d(Tx_{2n-1}, Tx_{2n-1})\} + cd(Tx_{2n-2}, Tx_{2n-1}) \\ &\le a \max\{d(Tx_{2n-2}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n})\} + bd(Tx_{2n-2}, Tx_{2n}) \\ &+ cd(Tx_{2n-2}, Tx_{2n-1}) \\ &\le a \max\{d(Tx_{2n-2}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n})\} \\ &+ b[d(Tx_{2n-2}, Tx_{2n-1}) + d(Tx_{2n-1}, Tx_{2n})] + cd(Tx_{2n-2}, Tx_{2n-1}) \end{aligned}$$

Again there are two cases,

Case-(III)

$$d(Tx_{2n-1}, Tx_{2n}) \le (a + b + c)d(Tx_{2n-2}, Tx_{2n-1}) + bd(Tx_{2n-1}, Tx_{2n})$$

$$\Rightarrow d(Tx_{2n-1}, Tx_{2n}) \le \frac{(a+b+c)}{(1-b)} d(Tx_{2n-2}, Tx_{2n-1}).$$

Case-(IV)

$$d(Tx_{2n-1}, Tx_{2n}) \le \frac{(b+c)}{[1-(a+b)]} d(Tx_{2n-2}, Tx_{2n-1})$$

Hence from all the cases, we conclude that,

$$d(Tx_{n+1}, Tx_n) \le kd(Tx_{n-1}, Tx_n) \le k^n d(Tx_1, Tx_0)$$

where $k = \max\left\{\frac{b+c}{1-(a+b)}, \frac{a+b+c}{1-b}\right\} < 1$ as $a + 2b + c < 1$, and for all $n \ge 0$

Hence for all $m \ge n \ge 0$

$$\begin{aligned} d(Tx_m, Tx_n) &\le \sum_{i=1}^{m-1} d(Tx_i, Tx_i + 1) \\ &\le \sum_{i=1}^{m-1} k^i d(Tx_1, Tx_0) \end{aligned}$$

$$\leq \frac{k^n}{1-k} d(Tx_1, Tx_0)$$

Then $d(Tx_m, Tx_n) \to 0$ as $m, n \to \infty$.

i.e. $\{Tx_n\}$ is a Cauchy sequence. Since $\overline{T(K)}$ is complete, then $\{Tx_n\}$ converges to some $z \in \overline{T(K)}$, and by the definition of $\{Tx_n\}$, we obtain that

$$\lim_{n\to\infty} gx_{2n} = z = \lim_{n\to\infty} fx_{2n+1}$$

Hence there exists $u, v \in K$ such that $fu = z = gv$. (Since $\overline{T(K)} \subseteq f(K) \cap g(K)$).

Let $\varepsilon$ be any positive number and $N$ a large number such that for any $n > N$

$$d(z, gx_{2n}) < \varepsilon, \; d(Tx_n, z) < \varepsilon, \; d(fx_{2n+1}, z) < \varepsilon$$

Then,

$$\begin{aligned} d(Tu, z) - \varepsilon \leq d(Tu, Tx_{2n}) & \\ & a \max\{d(fu, Tu), d(gx_{2n}, Tx_{2n})\} \\ & + b \min\{d(fu, Tx_{2n}), d(gx_{2n}, Tu)\} + cd(fu, gx_{2n}) \\ & \leq a \max\{d(Tu, z), d(gx_{2n}, z) + d(z, Tx_{2n})\} \\ & + b \min \{d(z, Tx_{2n}), d(gx_{2n}, z) - d(Tu, z)\} \\ & + c[d(fu, z) + d(z, gx_{2n})] \\ & \leq a \max\{d(Tu, z), 2\varepsilon\} + b\{\varepsilon, 0\} + c\varepsilon \end{aligned}$$

Now, there are two cases,

Case-I :

$$d(Tu, z) - \varepsilon \leq ad(Tu, z) + c\varepsilon \text{ implies that } \frac{(1+c)\varepsilon}{1-a} \geq d(Tu, z)$$

Case-II:

$$(2a + c\varepsilon) \geq d(Tu, z) - \varepsilon \text{ implies that } (1 + 2a + c)\varepsilon \geq d(Tu, z)$$

Since $\varepsilon$ be arbitrary positive number, we have $Tu = z$ i.e. we have proved that $u \in C(T, f)$.

Similarly, we can prove that $v \in C(T, g)$ and the first part of the theorem is proved. Next to prove second part, since $(T, f)$, $(T, g)$ are weakly compatible and $Tu = fu = z = Tv = gv$, then

$$gz = gTv = Tgv = Tz = Tfu = fTu = fz$$

Now, we prove that $z$ is a common fixed point of $T, f, g$. Suppose that $z \neq Tz$, then

$$\begin{aligned} d(z, Tz) = d(Tu, Tz) &\leq a \max\{d(fu, Tu), d(gz, Tz)\} \\ &\quad + b \min\{d(fu, Tz), d(gz, Tu)\} + cd(fu, gz) \\ &\leq a \max\{0, 0\} + b \min\{d(z, Tz), d(z, Tz)\} \\ &\quad + c\{d(fu, Tz) + d(gz, Tu)\} \\ &\leq (b + c)d(z, Tz) \end{aligned}$$

Hence $z \in F(T) \cap F(f) \cap F(g)$.

**Corollary 2.2.** *Let $K$ be a subset of a metric space $(E, d)$ and $T, f, g : K \to K$ are there mappings with $\overline{T(K)} \subseteq f(K) \cap g(K)$. Suppose that $\overline{T(K)}$ is complete, and $T$ is a generalized $(f, g)$-contraction with constants $a, b, c \in (0, 1)$ and $a + 2b + c < 1$. Then neither $C(T, f)$ nor $C(T, g)$ is empty. Moreover if in addition both $(T, f)$ and $(T, g)$ are weakly compatible, then $F(T) \cap F(f) \cap F(g)$ is singleton.*

Next we define generalized $(f, g)$-nenexpansive mapping as follows: let $K$ be a nonempty $q$-starshaped subset of a normed space $E$. A mapping $T : K \to K$ is called to be generalized $(f, g)$-nenexpansive if for all $x, y \in K$

$$\begin{aligned} \|Tx - Ty\| &\leq a \max\{d(fx, [Tx, q]), d(gy, [Ty, q])\} \\ &\quad + b \min\{d(fx, [Ty, q]), d(gy, [Tx, q])\} + c\| fx - gy \| \qquad \text{...(2.2)} \end{aligned}$$

with $a + 2b + c = 1$ where $a, b, c \in (0, 1)$

We obtain the following result in a normed space $E$.

**Theorem 2.3.** *Let $K$ be a nonempty q-starshaped subset of a normed space $E$, and let $T, f, g : K \to K$ be three continuous mappings and $T$ be a generalized $(f, g)$-nonexpansive mapping.*

*Suppose that both* $(T, f)$ *and* $(T, g)$ *are* $C_q$*-commuting, and both* $f$ *and* $g$ *are* $q$*-affine. If* $\overline{T(K)}$ *is compact subset of* $f(K) \cap g(K)$*, then* $F(T) \cap F(f) \cap F(g) \neq \phi$.

**Proof.** Let $\{\lambda_n\}$ be a strictly decreasing sequence in (0, 1) such that $\lim_{n\to\infty} \lambda_n = 1$. For each $n$, let $T_n$ be a mapping defined by

$$T_n x = (1 - \lambda_n)q + \lambda_n Tx, \text{ for all } x \in K$$

Then for all $n$, $\overline{T(K)} \subset f(K) \cap g(K)$ by $q$-starshapedness of $K$ and $q$-affiness of $f$ and $g$.

Thus for all $x, y \in K$

$$\begin{aligned}\|T_n x - T_n y\| &\leq \lambda_n \|Tx - Ty\| \\ &\leq \lambda_n [a \max\{d(fx, [Tx, q]), d(gy, [Ty, q])\} \\ &+ b \min\{d(fx, [Ty, q]), d(gy, [Tx, q])\} + c\|fx - gy\|] \\ &\leq \lambda_n [a \max\{\|fx - T_n x\|, \|gy - T_n y\|\} \\ &+ b \min\{\|fx - T_n y\|, \|gy - T_n x\|\} + c\|fx - gy\|]\end{aligned}$$

Then $T_n$, $f$, $g$ satisfy (2.2) with $\lambda_n \in (0, 1)$ and $a + 2b + c = 1$. Note that $(T, f)$ and $(T, g)$ are $C_q$-commuting and both $f$ and $g$ are $q$-affine, then $q \in F(f) \cap F(g)$ [1]. If $T_n x = fx = gx$, we have

$$T_n fx = (1 - \lambda_n)q + \lambda_n Tfx = (1 - \lambda_n)fq + \lambda_n fTx = f((1 - \lambda_n)q + \lambda_n Tx) = fT_n x$$

Namely, $(T_n, f)$ is weakly compatible. Similarly, we can prove that $(T_n, g)$ is weakly compatible. As $\overline{T(K)}$ is compact, then $\overline{T(K)}$ is complete [6, 8]. It follows from Corollary (2.2) that for each $n$, there exists a unique $x_n \in K$ such that

$$x_n = fx_n = gx_n = \lambda_n Tx_n + (1 - \lambda_n)q \qquad \text{...(2.3)}$$

It follows from the compactness of $\overline{T(K)}$ that there exists a sequence $\{x_{ni}\} \subset \{x_n\}$ and $z \in K$ such that $Tx_{ni} \to z \in \overline{T(K)}$.

Thus from (2.3),

$$x_{ni} = fx_{ni} = gx_{ni} = \lambda_{ni} Tx_{ni} + (1 - \lambda_{ni})q \to z \qquad \text{...(2.4)}$$

as $i \to \infty$

The continuity of $T$, $f$ and $g$ implies that $Tx_{ni} \to Tz$, $fx_{ni} \to fz$, and $gx_{ni} \to gz$ respectively. Hence from (2.4), we get $z = Tz = fz = gz$.

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# A NEW METHOD OF SOLVING SYSTEMS OF LINEAR FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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**ABSTRACT :** A new method of finding the solutions of systems of linear first-order ordinary differential equations with constant coefficients has been discussed.

**Key words :** Systems of linear first-order ordinary differential equations (SODEs), Systems of linear algebraic equations, Linear rth-order ordinary differential equation.
**AMS Classification.** 34B

## 1. INTRODUCTION

A system $S$ of linear first-order ordinary differential equations (SODEs) in $n$ unknows consists of $m$ equations of the form

$$S: M_i[x] = \sum_{j=1}^{n} l_{ij} x_j^{\prime}(t) - \sum_{j=1}^{n} a_{ij} x_j(t) = b_i(t) \qquad \text{...(1.1)}$$

where $t \in I$ (interval), $' \equiv \frac{d}{dt}$, $x = x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$, $l_{ij}$, $a_{ij} \in C$ (set of complex numbers) ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$), $b_1, b_2, ..., b_n$ are complex-valued functions on I, $A^T$ denotes the transpose of the matrix $A$.

The usual methods of finding the solutions of the SODEs (1.1) are stated below for comparison with the method to be presented:

(1) Laplace transforms are employed to each equation of the given SODEs (1.1) to derive $m$ linear algebraic equations in the $n$ unknows $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$, where $\hat{x}_i$ denotes the Laplace transform of $x_i$, $i = 1, 2, ..., n$. The solutions of these algebraic equations lead to the required solutions of the given SODEs (1.1), through inverse Laplace transforms.

(2) By differentiating the equations of the SODEs (1.1) requisite number of times and eliminating $n - 1$ of the $n$ dependent variables $x_1, x_2, ..., x_n$ , an rth-order ODE ($r$ = 2, 3, ...) is derived in the remaining dependent variable, $x_j$ say. Using for $x_j$ the solutions of this rth-order ODE, the given SODEs (1.1) is reduced to another SODEs of first-order in $n - 1$ variables. Repetitions of this process requisite number of times yield the required solutions of the given SODEs (1.1).

(3) Sometimes, by inspection, a linear combination of the equations of the given SODEs (1.1) can be so found that the resulting ordinary differential equation becomes solvable, leading to a linear algebraic equation in the $n$ unknows $x_1, x_2, ..., x_n$. If $k$ linearly independent such linear combinations of the $m$ equations of the given SODEs (1.1) can be found, the solutions of the corresponding $k$ algebraic equations in the $n$ unknowns $x_1, x_2, .., x_n$ will then help to find the required solutions of the given SODEs (1.1).

Here are some comments on the three methods described above:

The method (1) is not applicable always as the inverse Laplace transforms, required therein, may not be easily available.

The method (2) is obviously a very lengthy process. Further, the success of the method depends upon the success of finding the solutions of the rth-order ODE derived here.

The main lacuna of the method (3) is the "inspection"—part of it.

The aim of this presentation is to exhibit a new method of finding the solutions of the SODEs (1.1), by determining suitable linear combinations of the equations of the given system, as referred to in (3) above, of course, not by "inspection", but by following a systematic algorithm.

§2 deals with the main idea of solving SODEs with constant coefficients.

§3 deals with the case $m = n$. §4 gives the algorithm; §5 gives an illustration while §6 deals with some special cases. Some remarks have been given in §7.

## 2. THE MAIN IDEA

If $\lambda_i$ ($\in C$), $i = 1, 2, ..., m$ can be so found that

$$\sum_{i=1}^{m} \lambda_i M_i[x] \equiv \left( \sum_{i=1}^{n} v_i x_i \right)' - k\left( \sum_{i=1}^{n} v_i x_i \right), \qquad ...(2.1)$$

for some $k$, $v_i \in C$ ($i = 1, 2, ..., n$), then using (1.1) one obtains from (2.1) the following differential equation

$$\left( \sum_{i=1}^{n} v_i x_i \right)' - k \sum_{i=1}^{n} v_i x_i = \sum_{j=1}^{m} \lambda_j b_j(t), \; (t \in I), \qquad ...(2.2)$$

It is noted that (2.2) is a linear first-order ODE, and hence is solvable. Actually (2.2) yields

$$\sum_{i=1}^{n} v_i x_i = \exp(kt) \left[ C + \int \exp(-kt) \sum_{j=1}^{m} \lambda_j b_j(t) dt \right], \qquad ...(2.3)$$

where $C$ is the parameter of integration.

Using (1.1) in (2.1) one gets

$$\sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} l_{ij} x'_j - \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} v_j x'_j - k \sum_{j=1}^{n} v_j x_j \qquad ...(2.4)$$

Equating the coefficients of $x'_j$, $x_j$ ($j = 1, 2, ..., n$) from the two sides of (2 4) one obtains

$$\sum_{i=1}^{m} \lambda_i l_{ij} = v_j, \qquad ...(2.5a)$$

$$\sum_{i=1}^{m} \lambda_i a_{ij} = k v_j, \qquad ...(2.5b)$$

for $j = 1, 2, ..., n$. From (2.5a) and (2.5b) one gets

$$\sum_{i=1}^{m} \lambda_i \left( a_{ij} - k l_{ij} \right) = 0 \qquad (j = 1, 2, ..., n). \qquad .(2.6)$$

The $n$ linear algebraic equations (2.6) can now be solved for $\lambda_1, \lambda_2, ..., \lambda_m$, in the usual way.

Having obtained $\lambda_1, \lambda_2, \ldots, \lambda_m$, one gets $\nu_1, \nu_2, \ldots, \nu_n$ from (2.5a), and so the algebraic equation (2.3) is obtained. The number of such algebraic equations depends on the number of distinct complex numbers $k$ satisfying (2.6).

## 3. THE CASE $m = n$

If, in particular, $m = n$, the $n$ linear homogeneous algebraic equations in (2.6) determine a nontrivial solution for $\lambda_1, \lambda_2, \ldots, \lambda_n$ provided

$$\begin{vmatrix} a_{11} - kl_{11} & a_{21} - kl_{21} & \ldots & a_{nl} - kl_{nl} \\ a_{12} - kl_{12} & a_{22} - kl_{22} & \ldots & a_{n2} - kl_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n} - kl_{1n} & a_{2n} - kl_{2n} & \ldots & a_{nn} - kl_{nn} \end{vmatrix} = 0 \qquad \ldots(3.1)$$

The equation (3.1) is, in general, an $n$th-degree polynomial equation in $k$. So (3.1) possesses $n$ roots, $k_1, k_2, \ldots, k_n$, corresponding to each of which, the linear homogeneous algebraic equations (2.6) possesses a nontrivial solution for $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let the solution of (2.6) for $k = k_r$ be $\lambda_{1r}, \lambda_{2r}, \ldots, \lambda_{nr}$ and $\nu_{ir} = \sum_{i=1}^{n} \lambda_{ir} l_{ij}$, $r = 1, 2, \ldots, n$. Then, from (2.3) one obtains

$$\sum_{i=1}^{n} \nu_{ir} x_i = \exp(k_r t)\left[C_r + \int \exp(-k_r t) \sum_{j=1}^{n} \lambda_{jr} b_j(t)\, dt\right], \qquad \ldots(3.2)$$

where $C_r$ ($\in C$) is the corresponding parameter of integration and $r = 1, 2, \ldots, n$. Solving the $n$ linear algebraic equations of (3.2), the required solutions of the SODEs (1.1), with $m = n$, are obtained.

## 4. THE ALGORITHM

The steps to be followed for solving the SODEs (1.1) with $m = n$ are the following:

*Step I* : Solve the polynomial equation (3.1).

*Step II* : For each root $k_r$ $(r = 1, 2, .., n)$ of (3.1), solve the system of algebraic equations (2.6) with $k = k_r$. Let the corresponding solution be denoted by $\lambda_{1r}, \lambda_{2r}, ..., \lambda_{nr}$.

*Step III* : Solve the linear ODE (2.2) with $v_j = v_{jr}$, $j = 1, 2, ..., n$, to obtain (3.2).

*Step IV* : Solve the system of $n$ algebraic equations (3.2) for $x_1, x_2, ..., x_n$.

The solutions obtained for $x_1, x_2, ..., x_n$ are the required solutions of the SODEs (1.1) with $m = n$.

*N.B.* 1. : The algebraic equations in (2.6) being homogeneous, it is enough to determine $\lambda_1 : \lambda_2 : ... : \lambda_n$.

*N.B. 2.* : If the degree of the polynominal equations (3.1) is less than $n$, then the $n$ expressions $\sum_{i=1}^{n} l_{ij} x'_j$, $i = 1, 2, ..., n$, are linearly dependent. So the rank of the matrix $L = (l_{ij})$ is less than $n$. If rank $L = r$, then $n - r$ algebraic equations in $x_1, x_2, ..., x_n$ can be derived from the given set of SODEs (1.1) without integration. Using these $n - r$ algebraic equations, the given SODEs (1.1) in $n$ unknowns can be reduced to a system in $r$ unknows, which can be handled following the algorithm given in §4.

## 5. AN ILLUSTRATION

**Example 1 :** $2x'_1 - 2x'_2 - 3x_1 = t,$ ... (ia)

$2x'_1 + 2x'_2 + 3x_1 + 8x_2 = 2$ ...(ib)

where $t \in \mathbb{R}$ (set of real numbers).

In this case (3.1) turns out to be $k^2 - 2k - 3 = 0.$ ...(iii)

For the root 3 of (ii), the two equations of (2.6) are identical, viz

$$3\lambda_1 + \lambda_2 = 0.$$

Subtracting 3 times (ib) from (ia), one gets

$$-4x'_1 - 8x'_2 - 12x_1 - 24x_2 = t - 6. \qquad ...(iii)$$

The solution of the linear ODE (iii) is given to be

$$x_1 + 2x_2 = \frac{19}{36} - \frac{t}{12} + C_1 e^{-3t}, \qquad \text{.. (iv)}$$

where $C_1$ ($\in C$) is the parameter of integration.

For the root $-1$ of (ii), the two equations of (2.6) are identical, viz.

$$\lambda_1 - 5\lambda_2 = 0.$$

Adding (ib) to 5 times (ia) one gets

$$12x'_1 - 8x'_2 - 12x_1 + 8x_2 = 5t + 2, \qquad \text{...(v)}$$

whence, on integration, one obtains

$$12x_1 - 8x_2 = -7 - 5t + C_2 e^t, \qquad \text{...(vi)}$$

where $C_2$ ($\in C$) is the parameter of integration.

The required solutions of (ia) – (ib) are then obtained by solving the algebraic equations (iv) and (vi).

## 6. SOME SPECIAL CASES

Two examples are cited below where the degree of the corresponding polynomial equation (3.1) is less than $n$.

**Example 2:**

$$\left.\begin{aligned} x'_1 + x'_2 - x_1 + x_2 &= 0, && \text{...(ia)} \\ 2x'_1 + 2x'_2 - 2x_1 + 2x_2 &= t, && \text{...(ib)} \end{aligned}\right\} (t \in R)$$

Here (3.1) becomes $8k = 0$ : its degree $= 1 < 2$ ($= n$). ...(ii)

Using $k = 0$ in the corresponding (2.6) one gets $-\lambda_1 + 2\lambda_2 = 0$

Taking $\lambda_1 = 2$, $\lambda_2 = 1$, one finds that 2.(ia) + (ib) gives

$$4x'_1 + 4x'_2 = t.$$

Hence, on integration, one gets

$$x_1 + x_2 = \frac{1}{8}t^2 + C_1, \ (C_1 : \text{real parameter}). \qquad \text{...(iii)}$$

Notably, the other algebraic equation is obtained by subtracting 2.(ia) from (ib) as

$$4x_1 - 4x_2 = t. \qquad \text{...(iv)}$$

(iii) and (iv) determine the required solutions of (ia) – (ib).

**Example 3 :**

$$\left.\begin{aligned} x'_1 + x'_2 + 3x_1 + x_2 &= e^t, \qquad \text{...(ia)} \\ x'_1 + x'_2 + x_1 - x_2 &= t, \qquad \text{...(ib)} \end{aligned}\right\} \ (t \in R)$$

In this case the determinant in (3.1) becomes

$$\begin{vmatrix} 3-k & 1-k \\ 1-k & -1-k \end{vmatrix} = \begin{vmatrix} 2 & 1-k \\ 2 & -1-k \end{vmatrix} = 4 \neq 0.$$

This implies that two linear algebraic equations in $x_1, x_2$ can be obtained from (ia), (ib) without integration. In fact, subtracting (ib) from (ia) one gets

$$2x_1 + 2x_2 = e^t - t. \qquad \text{...(ii)}$$

Eliminating one of $x_1$, $x_2$, say $x_2$, from (ii) and one of (ia), (ib), say (ib), one obtains

$$x_1' + \frac{1}{2}\left(e^t - t - 2x_1\right)' + x_1 - \frac{1}{2}\left(e^t - t - 2x_1\right) = t$$

or, $$x_1 = \frac{1}{4}(t+1). \qquad \text{.. (iii)}$$

(ii) and (iii) determine the required solution of (ia) – (ib), which does not contain any parameter of integration, as no integration has been performed.

## 7. REMARKS

The method presented here of solving a system of $n$ linear first-order ordinary differential equations in $n$ unknowns comprises only four extremely simple steps (vide §4). The algorithm

also indicates the number of linear algebraic equations, derivable from the given system of ordinary differential equations without integration.

The extension of the method described above to systems of linear first-order ordinary differential equations with variable coefficients will be presented in a subsequent paper.

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# FIXED POINT THEOREM IN HILBERT SPACE FOR THREE MAPPING

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**ABSTRACT :** We find unique common random fixed point theorem using contractive condition for three continuous random operators defined on separable Hilbert space.

**Key words :** Separable Hilbert Space, random operator, fixed point.

**Mathematics Subject Classification.** 54H25, 47H10.

## 1. INTRODUCTION AND PRELIMINARY NOTES

In this paper, we construct a sequence of measurable functions and consider its convergence to the common unique random fixed point of three continuous random operators defined on a non-empty closed subset of a separable Hilbert space. Some of the recent literatures in random fixed point may be noted in [1, 2, 3, 4, 5].

In this paper, $(\Omega, \Sigma)$ denotes a measurable space, $H$ stands for a separable Hilbert space, and $C$ is a nonempty subset of $H$. A function $f : \Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset $B$ of $H$. A function $F : \Omega \times C \to C$ is said to be a random operator if $F(., x) : \Omega \to C$ is measurable for every $x \in C$. A measurable function $g : \Omega \to C$ is said to be a random fixed point of the random operator $F : \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$. A random operator $F : \Omega \times C \to C$ is to be continuous if for fixed $t \in \Omega$, $F(t, .) : C \to C$ is continuous.

**Theorem.** Let $C$ be a non-empty closed subset of a separable Hilbert space $H$. Let $A$, $B$ and $T$ be three continuous random operators defined on $C$ such that for $t \in \Omega$, $A(t, .)$, $B(t, .)$, $T(t, .) : C \to C$ satisfy condition

(i) $A(H) \cup B(H) \subset T(H)$,

(ii) $AT = TA$, $BT = TB$,

(iii) $$\| Ax - By \|^2 \le a \left[ \frac{\| Ty - Ax \|^2 \| By - Tx \|^2}{\| Ax - Ty \|^2 + \| By - Ty \|^2} \right].$$

$$+ \, b \, \| Ty - Ax \|^2 \left[ \frac{\| Ax - Tx \|^2 + \| By - Ty \|^2}{\| By - Tx \|^2 + \| Ax - Ty \|^2} \right]$$

$$+ \, c \, [\| Ty - Ax \|^2 + \| By - Tx \|^2] \text{ for all } x, y \in C \text{ and,}$$

where $a, b, c, d > 0$, $2a + b + 4c < 1$. Then $A$, $B$ and $T$ have unique common random fixed point.

**Proof:** Let the function $g_0 : \Omega \to C$ be arbitrary measurable function. By (i) there exist $g_1 : \Omega \to C$ such that $T(t, g_1(t)) = A(t, g_0(t))$ for $t \in \Omega$ and for this function $g_1 : \Omega \to C$, we can choose another function $g_2 : \Omega \to C$ such that $T(t, g_2(t)) = B(t, g_1(t))$ for $t \in \Omega$ and so on. Inductively we can define

$$T(t, g_{2n+1}(t)) = A(t, g_{2n}(t)),$$

and

$$T(t, g_{2n+2}(t)) = B(t, g_{2n+1}(t)) \qquad \ldots(1)$$

for $t \in \Omega$ and $n = 0, 1, 2, 3, \ldots$

for condition (ii) we have for $t \in \Omega$

$$\| T(t, g_{2n+1}(t)) - T(t, g_{2n+2}(t)) \|^2 = \| A(t, g_{2n}(t)) - B(t, g_{2n+1}(t)) \|^2$$

$$\le a \left[ \frac{\| T(t, g_{2n+1}(t)) - A(t, g_{2n}(t)) \|^2 \, \| B(t, g_{2n+1}(t)) - T(t, g_{2n}(t)) \|^2}{\| A(t, g_{2n}(t)) - T(t, g_{2n+1}(t)) \|^2 + \| B(t, g_{2n+1}(t)) - T(t, g_{2n+1}(t)) \|^2} \right]$$

$$+ \, b \, \| T(t, g_{2n+1}(t)) - A(t, g_{2n}(t)) \|^2$$

$$\left[ \frac{\| A(t, g_{2n}(t)) - T(t, g_{2n}(t)) \|^2 + \| B(t, g_{2n+1}(t)) - T(t, g_{2n+1}(t)) \|^2}{\| B(t, g_{2n+1}(t)) - T(t, g_{2n}(t)) \|^2 + \| A(t, g_{2n}(t)) - T(t, g_{2n+1}(t)) \|^2} \right]$$

$$+ c\,[\|T(t, g_{2n+1}(t)) - A(t, g_{2n}(t))\|^2 + \|B(t, g_{2n+1}(t)) - T(t, g_{2n}(t))\|^2],$$

$$\leq a\left[\frac{\|T(t,g_{2n+1}(t)) - T(t,g_{2n+1}(t))\|^2 \, \|T(t,g_{2n+2}(t)) - T(t,g_{2n}(t))\|^2}{\|T(t,g_{2n+1}(t)) - T(t,g_{2n+1}(t))\|^2 + \|T(t,g_{2n+2}(t)) - T(t,g_{2n+1}(t))\|^2}\right]$$

$$+ b\,\|T(t, g_{2n+1}(t)) - T(t, g_{2n+1}(t))\|^2$$

$$\left[\frac{\|T(t,g_{2n+1}(t)) - T(t,g_{2n}(t))\|^2 + \|T(t,g_{2n+2}(t)) - T(t,g_{2n+1}(t))\|^2}{\|T(t,g_{2n+2}(t)) - T(t,g_{2n}(t)\|^2 + \|T(t,g_{2n+1}(t)) - T(t,g_{2n+1}(t)\|^2}\right]$$

$$+ c[\|T(t, g_{2n+1}(t)) - T(t, g_{2n+1}(t))\|^2 + \|T(t, g_{2n+2}(t)) - T(t, g_{2n}(t))\|^2]$$

$$\Rightarrow (1 - 2c)\ \|T(t, g_{2n+1}(t)) - T(t, g_{2n+2}(t))\|^2 \leq 2c\|T(t, g_{2n+1}(t)) - T(t, g_{2n}(t))\|^2$$

$$\Rightarrow \|T(t, g_{2n+1}(t)) - T(t, g_{2n+2}(t))\|^2 \leq \frac{2c}{(1-2c)}\ \|T(t, g_{2n+1}(t)) - T(t, g_{2n}(t))\|^2$$

$$\Rightarrow \|T(t, g_{2n+1}(t)) - T(t, g_{2n+2}(t))\| \leq k_1\|T(t, g_{2n+1}(t)) - T(t, g_{2n}(t))\| \qquad ...(2)$$

Where $\left(\frac{2c}{1-2c}\right)^{\frac{1}{2}} = k_1 < 1$, since $c < 1$.

Similarly $\|T(t, g_{2n+1}(t)) - T(t, g_{2n}(t))\|^2 = \|A(t, g_{2n}(t)) - B(t, g_{2n-1}(t))\|^2$

$$\leq a\left[\frac{\|T(t,g_{2n-1}(t)) - A(t,g_{2n}(t))\|^2 \, \|B(t,g_{2n-1}(t)) - T(t,g_{2n}(t))\|^2}{\|A(t,g_{2n}(t)) - T(t,g_{2n-1}(t))\|^2 + \|B(t,g_{2n-1}(t)) - T(t,g_{2n-1}(t))\|^2}\right]$$

$$+ b\|T(t, g_{2n-1}(t)) - A(t, g_{2n}(t))\|^2$$

$$\left[\frac{\|A(t,g_{2n}(t)) - T(t,g_{2n}(t))\|^2 + \|B(t,g_{2n-1}(t)) - T(t,g_{2n-1}(t))\|^2}{\|B(t,g_{2n-1}(t)) - T(t,g_{2n}(t)\|^2 + \|A(t,g_{2n}(t)) - T(t,g_{2n-1}(t))\|^2}\right]$$

$$+ c[\|T(t, g_{2n-1}(t)) - A(t, g_{2n}(t))\|^2 + \|B(t, g_{2n-1}(t)) - T(t, g_{2n}(t))\|^2],$$

$$\leq a\left[\frac{\|T(t,g_{2n-1}(t))-T(t,g_{2n+1}(t))\|^2\,\|T(t,g_{2n}(t))-T(t,g_{2n}(t))\|^2}{\|T(t,g_{2n+1}(t))-T(t,g_{2n-1}(t))\|^2+\|T(t,g_{2n}(t))-T(t,g_{2n-1}(t))\|^2}\right]$$

$$+\ b\|T(t,\ g_{2n-1}(t))\ -\ T(t,\ g_{2n+1}(t))\|^2$$

$$\left[\frac{\|T(t,g_{2n+1}(t))-T(t,g_{2n}(t))\|^2+\|T(t,g_{2n}(t))-T(t,g_{2n-1}(t))\|^2}{\|T(t,g_{2n}(t))-T(t,g_{2n}(t))\|^2+\|T(t,g_{2n+1}(t))-T(t,g_{2n-1}(t))\|^2}\right]$$

$$+\ c[\|T(t,\ g_{2n-1}(t))\ -\ T(t,\ g_{2n+1}(t))\|^2\ +\ \|T(t,\ g_{2n}(t))\ -\ T(t,\ g_{2n}(t))\|^2],$$

$$\Rightarrow (1-b-2c)\,\|T(t,\ g_{2n+1}(t))\ -\ T(t,\ g_{2n}(t))\|^2$$

$$\leq (b+2c)\ \|T(t,\ g_{2n-1}(t))\ -\ T(t,\ g_{2n}(t))\|^2$$

$$\Rightarrow \|T(t,\ g_{2n+1}(t))\ -\ T(t,\ g_{2n}(t))\|^2 \leq \frac{b+2c}{(1-b-2c)}\ \|T(t,\ g_{2n-1}(t))\ -\ T(t,\ g_{2n}(t))\|^2$$

$$\Rightarrow \|T(t,\ g_{2n+1}(t))\ -\ T(t,\ g_{2n}(t))\| \leq k_2\|T(t,\ g_{2n-1}(t))\ -\ T(t,\ g_{2n}(t))\| \qquad \ldots(3)$$

where $\left(\frac{2c}{1-b-2c}\right)^{\frac{1}{2}} = k_2 < 1$, Since $1 + b + 4c < 1$.

The inequality (2) and (3) jointly imply that

$\|T(t,\ g_{2n+1}(t))\ -\ T(t,\ g_{2n+2}(t)\| \leq k\,\|T(t,\ g_{2n+1}(t))\ -\ T(t,\ g_{2n}(t))\|$

where $k = \max\{k_1,\ k_2\} < 1$

$\Rightarrow \|T(t,\ g_n(t))\ -\ T(t,\ g_{n+1}(t))\| \leq k^n\|T(t,\ g_1(t))\ -\ T(t,\ g_0(t))\|$

for all $n = 1, 2, \ldots$

Now we shall prove that $t \in \Omega$, $T(t,\ g_n(t))$ is a Cauchy sequence for this every positive integer $p$ we have

$\|T(t,\ g_n(t))\ -\ T(t,\ g_{n+p}(t))\|$

$$= \|T(t, g_n(t)) - T(t, g_{n+1}(t)) + T(t, g_{n+1}(t)) - \dots\dots\dots + T(t, g_{n+p-1}(t)) - T(t, g_{n+p}(t))\|$$

$$\leq \|T(t, g_n(t)) - T(t, g_{n+1}(t))\| + \|T(t, g_{n+1}(t)) - T(t, g_{n+2}(t))\| + \dots\dots\|T(t, g_{n+p-1}(t)) - T(t, g_{n+p}(t))\|$$

$$\leq [k^n + k^{n+1} + \dots\dots\dots + k^{n+p-1}]\|T(t, g_1(t)) - T(t, g_0(t))\|$$

$$= k^n[1 + k + k^2 + \dots\dots\dots + k^{p-1}]\ \|T(t, g_1(t)) - T(t, g_0(t))\|$$

$$< \frac{k^n}{1-k}\|T(t, g_1(t)) - T(t, g_0(t))\|.$$

This implies $\|T(t, g_n(t)) - T(t, g_{n+p}(t))\| \to 0$ as $n \to \infty$ for $t \in \Omega$ ...(4)

equation (4), is a Cauchy sequence and hence is convergent in closed subset $C$ of Hilbert space $H$. there exist $g(t)$ such that

$$T(t, g_n(t)) \to g(t),$$

$$A(t, g_n(t)) \to g(t),$$

$$B(t, g_n(t)) \to g(t),$$

from (i). Since $A$, $B$ and $T$ are continuous operators and $AT = TA$ and $BT = TB$.

$$A(t, T(t, g_n(t))) \to A(t, g(t))$$

$$B(t, T(t, g_n(t))) \to B(t, g(t))$$

$$T(t, A(t, g_n(t))) \to T(t, g(t))$$

$$T(t, B(t, g_n(t))) \to T(t, g(t))$$

Therefore from (i) $A(t, g(t)) = T(t, g(t)) = B(t, g(t))$ for $t \in \Omega$, ...(5)

Existence of random fixed point: Consider for $t \in \Omega$

$$\|A(t, g(t)) - g(t)\|^2 = \|A(t, g(t)) - B(t, g_{2n+1}(t)) + B(t, g_{2n+1}(t)) - g(t)\|^2$$

$\leq 2\|A(t, g(t)) - B(t, g_{2n+1}(t))\|^2 + 2\|B(t, g_{2n+1}(t)) - g(t)\|^2$

$$\leq 2a\left[\frac{\|T(t,g_{2n+1}(t)) - A(t,g(t))\|^2\|B(t,g_{2n+1}(t)) - T(t,g(t))\|^2}{\|A(t,g(t)) - T(t,g_{2n+1}(t))\|^2 + \|B(t,g_{2n+1}(t)) - T(t,g_{2n+1}(t))\|^2}\right]$$

$$+ 2b\|T(t, g_{2n+1}(t)) - A(t, g(t))\|^2\left[\frac{\|A(t,g(t)) - T(t,g(t))\|^2\|B(t,g_{2n+1}(t)) - T(t,g_{2n+1}(t))\|^2}{\|B(t,g_{2n+1}(t)) - T(t,g(t))\|^2 + \|A(t,g(t)) - T(t,g_{2n+1}(t))\|^2}\right]$$

$+ 2c[\|T(t, g_{2n+1}(t)) - A(t, g(t))\|^2 + \|B(t, g_{2n+1}(t)) - T(t, g(t))\|^2]$

$+ 2\|B(t, g_{2n+1}(t)) - g(t)\|^2$ when $n \to \infty$

$$\leq 2a\left[\frac{\|g(t) - A(t,g(t))\|^2\|g(t) - A(t,g(t))\|^2}{\|A(t,g(t)) - g(t)\|^2 + \|g(t) - g(t)\|^2}\right]$$

$$+2b\ \|g(t) - A(t, g(t))\|^2\left[\frac{\|A(t,g(t)) - A(t,g(t))\|^2 + \|g(t) - g(t)\|^2}{\|g(t) - A(t,g(t))\|^2 + \|A(t,g(t)) - g(t)\|^2}\right]$$

$+ 2c[\|g(t) - A(t, g(t))\|^2 + \|g(t) - A(t, g(t))\|^2 + 2\|g(t) - g(t)\|^2$.

This implies $(1 - 2a - 4c)\|A(t, g(t)) - g(t)\|^2 \leq 0$.

$\Rightarrow A(t, g(t)) = g(t)$ for all $t \in \Omega$.

Similarly $B(t, g(t)) = g(t) = T(t, g(t))$ for all $t \in \Omega$.

This implies $g : \Omega \to C$ is a common random fixed point of $A$, $B$ and $T$.

## 2. UNIQUENESS

Let $h : \Omega \to C$ be another random fixed point common to $A$, $B$ and $T$ that is, for $t \in \Omega$,

$$A(t, h(t)) \to h(t),\ B(t, h(t)) \to h(t)$$

$$T(t, h(t)) \to h(t),$$

Then for $t \in \Omega$,

$$\|g(t) - h(t)\|^2 = \|A(t, g(t)) - B(t, h(t))\|^2$$

$$\leq a \left[ \frac{\|T(t,h(t)) - A(t,g(t))\|^2 + \|B(t,h(t)) - T(t,g(t))\|^2}{\|A(t,g(t)) - T(t,h(t))\|^2 + \|B(t,h(t)) - T(t,h(t))\|^2} \right]$$

$$+ b\|T(t, h(t)) - A(t, g(t))\|^2 \left[ \frac{\|A(t,g(t)) - T(t,g(t))\|^2 + \|B(t,h(t)) - T(t,h(t))\|^2}{\|B(t,h(t)) - T(t,g(t))\|^2 + \|A(t,g(t)) - T(t,h(t))\|^2} \right]$$

$$+ c[\|T(t, h(t)) - A(t, g(t))\|^2 + \|B(t, h(t)) - T(t, g(t))\|^2]$$

$\Rightarrow (1 - a - 2c)\ \|g(t) - h(t)\|^2 \leq 0$. This implies $g(t) = h(t)$, for all $t \in \Omega$,

Hence $g : \Omega \to C$ is a unique common fixed point in $A$, $B$ and $T$.

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# ON SOME FORMS OF (1, 2)*-CONTINUITY IN BITOPOLOGICAL SPACES

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**ABSTRACT :** The notions of (1,2)*-semi-continuity [43], (1, 2)*-precontinuity [3] and (1, 2)*-$\alpha$-continuity [3] between bitopological spaces are studied. We deduce the study of (1, 2)*-continuity forms between bitopological spaces to the study of $M$-continuity between $m$-spaces and obtain unified properties of these continuity by using the results established in [33] and [40].

**Key words :** Key words and phrases: $m$-structure, $m$-space. $mg$-closed, $M$-continuity, (1, 2)*-semi-continuity, bitopological space.

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## 1. INTRODUCTION

Semi-open sets, preopen sets, $\alpha$-open sets and $\beta$-open sets play an important role in the researches of generalizations of continuity in topological spaces and bitopological spaces. The notions of quasi-open sets [14], [46] or $\tau_{1,2}$-open sets [44] in bitopological spaces are in troduced and studied. The notions of $\tau_{1,2}$-open sets and $\tau_{1,2}$-continuity, (1, 2)*-semi-open sets and (1,2)*-semi-continuity, (1, 2)*-preopen sets and (1, 2)*-precontinuity, (1, 2)*-$\alpha$-sets and (1, 2)*-$\alpha$-continuity are introduced and studied in [45], [46], [43] and [14].

In [40] and [41], the present authors introduced and studied the notions of minimal structures, $m$-spaces, $m$-continuity and $M$-continuity.

The concept of generalized closed sets in topological spaces was introduced by Levine [20]. The notion of $g$-continuity is introduced and studied in [30], [7], [8] and other papers. Noiri [33] introduced the notion of $mg$-closed sets. Recently, in [11], [35], [37], the authors reduced the study of some continuity forms between bitopological spaces to the study of $m$-continuity and $M$-continuity between $m$-spaces.

In the present paper, we deduce the study of (1, 2)*-continuity forms between bitopological spaces to the study of $M$-continuity between $m$-spaces by generalizing some results established in [45], [44], [43] and [38].

## 2. PRELIMINARIES

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

**Definition 2.1** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be *semi-open* [19] (resp. *preopen* [25], *α-open* [31], *β-open* [1] or *semi-preopen* [4]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

The family of all semi-open (resp. preopen, α-open, β-open) sets in $(X, \tau)$ is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$).

**Definition 2.2** The complement of a semi-open (resp. preopen. α-open, β-open) set is said to be *semi-closed* [12] (resp. *preclosed* [25], *α-closed* [26], *semi-preclosed* [4]).

**Definition 2.3** The intersection of all semi-closed (resp. preclosed, α-closed, β-closed) sets of $X$ containing $A$ is called the *semi-closure* [12] (resp. *preclosure* [15], *α-closure* [26], *semi-preclosure* [4]) of $A$ and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$).

**Definition 2.4** The union of all semi-open (resp. preopen, α-open, β-open) sets of $X$ contained in $A$ is called the *semi-interior* (resp. *preinterior, α-interior, semi-preinterior)* of $A$ and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{spInt}(A)$).

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ denote topological spaces and $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ denote bitopological spaces.

## 3. MINIMAL STRUCTURES

**Definition 3.1** A subfamily $m_X$ of the power set $\mathcal{P}(X)$ of a nonempty set $X$ is called a *minimal structure* (briefly *m-structure)* on $X$ [40], [41] if $\emptyset \in m_X$ and $X \in m_X$.

By $(X, mx)$, we denote a nonempty subset $X$ with a minimal structure $m_X$ on $X$ and call

it an $m$-space. Each member of $m_X$ is said to be *$m_X$-open* (briefly $m$-open) and the complement of an $m_X$-open set is said to be *$m_X$-closed* (briefly $m$-closed).

**Remark 3.1** Let $(X, \tau)$ be a topological space. Then the families $\tau$, SO($X$), PO($X$), $\alpha(X)$, $\beta(X)$ and SPO($X$) are all $m$-structures on $X$.

**Definition 3.2** Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the *$m_X$-closure* of $A$ and the *$m_X$-interior* of $A$ are defined in [24] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset \mathrm{A}, U \in m_X\}$.

**Remark 3.2** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_X = \tau$ (resp. SO($X$), PO($X$), $\alpha(X)$, $\beta(X)$, SPO($X$)), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $_{\beta}\mathrm{Cl}(\mathrm{A})$, $\mathrm{spCl}(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$, $\mathrm{spInt}(A)$).

**Lemma 3.1** (Maki et al, [24]) *Let $X$ be a nonempty set and $m_X$ an m-structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:*

(1) $m\mathrm{Cl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = \mathrm{X} \setminus m\mathrm{Cl}(\mathrm{A})$,

(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$. $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

**Lemma 3.2** (Popa and Noiri [40]) *Let $(X, m_X)$ be an m-space and $A$ a subset of $X$. Then* $x \in \mathrm{mCl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing x.*

**Definition 3.3** An $m$-structure $m_X$ on a nonempty set $X$ is said to have *property* B [24] if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

**Remark 3.3** Let $(X, \tau)$ be a topological space. Then the families $\tau$, SO($X$), PO($X$), $\alpha(X)$, $\beta(X)$ and SPO($X$) have property $\mathcal{B}$.

**Lemma 3.3** (Popa and Noiri [42]) *Let $X$ be a nonempty set and $m_X$ an m-structure on $X$ having property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold:*

(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,

(2) $A$ *is $m_X$-closed if and only if* $\mathrm{mCl}(A) = A$,

(2) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ is *$m_X$-closed.*

**Definition 3.4** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be *quasi open* [14], [46] or *$\tau_1\tau_2$-open* [43] if $A = B \cup C$, where $B \in \tau_1$ and $C \in \tau_2$. The complement of a $\tau_1\tau_2$-open set is said to be *$\tau_1\tau_2$-closed.* $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed are simply denoted by $\tau_{1,2}$-open and $\tau_{1,2}$-closed, respectively.

In the following, the collection of all $\tau_{1,2}$-open sets of $X$ is denoted by $\tau_{1,2}O(X)$. For a subset $A$ of $X$, the $\tau_1\tau_2$-closure $\tau_1\tau_2\mathrm{Cl}(A)$ (simply $\tau_{1,2}\mathrm{Cl}(A)$) of $A$ and the $\tau_1\tau_2$-interior $\tau_1\tau_2$ Int(A) (simply $\tau_{1,2}\mathrm{Int}(A)$) of $A$ are defined as follows:

(1) $\tau_1\tau_2\mathrm{Cl}(A) = \cap\{F : A \subset F, X \setminus F \in \tau_{1,2}O(X)\}$,

(2) $\tau_1\tau_2\mathrm{Int}(A) = \cup\{U : U \subset A,\ U \in \tau_{1,2}O(X)\}$.

**Definition 3.5** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be

(1) *(1, 2)*-semi-open* [43], [44] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(A))$,

(2) *(1, 2)*-preopen* [43], [44] if $A \subset \tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A))$,

(3) *(1, 2)*-$\alpha$-open* [43], [44] if $A \subset \tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(A)))$,

(4) *(1, 2)*-semi-preopen* if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A)))$.

The complement of a (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-$\alpha$-open, (1, 2)*-semi-preopen) set is said to be (1, 2)*-semi-closed (resp. (1, 2)*-preclosed, (1, 2)*-$\alpha$-closed, (1, 2)*-semi-preclosed).

The family of all (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-$\alpha$-open, (1, 2)*-semi-preopen) sets is denoted by (1, 2)*SO($X$) (resp. (1, 2)*PO($X$), (1, 2)*$\alpha$($X$), (1, 2)*SPO($X$))

**Remark 3.4** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ a subset of $X$.

(1) The families $\tau_{1,2}O$(X), (1, 2)*SO($X$), (1, 2)*PO($X$), (1, 2)*$\alpha$($X$), and (1, 2)*SPO($X$) are all $m$-structures with property B.

(2) In the following, we denote by $m(\tau_1, \tau_2)$ (briefly $m_{1,2}$) each member of the above five families and **call it an $m$-structure determinded by** $\tau_1$ and $\tau_2$. Let $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. (1, 2)*SO($X$), (1, 2)*PO($X$), (1, 2)*$\alpha$($X$), (1, 2)*SPO($X$)), then we have (i) $m_{1,2}\mathrm{Cl}(A) = \tau_1\tau_2\mathrm{Cl}(A)$ (resp. (1. 2)*sCl($A$), (1, 2)*pCl($A$), (1.2)*$\alpha$Cl($A$), (1, 2)*spCl($A$)), (ii) $m_{1,2}\mathrm{Int}(A) = \tau_1\tau_2\mathrm{Int}(A)$ (resp. (1,2)*sInt($A$), (1, 2)*pInt($A$), (1, 2)*$\alpha$Int(A), (1, 2)*spInt($A$)).

(3) Since each one of $m(\tau_1, \tau_2)$ has property B, by Lemma 3.3 we have

(i) $A$ is $m(\tau_1, \tau_2)$-closed if and only if $m_{1,2}\mathrm{Cl}(A) = A$,

(ii) $A$ is $m(\tau_1, \tau_2)$-open if and only if $m_{1,2}\mathrm{Int}(A) = A$

for $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. (1, 2)*SO($X$), (1, 2)*PO($X$), (1,2)*$\alpha$($X$), (1, 2)*SPO(X)).

(4) By Lemma 3.2, we obtain the result established in Proposition 2.2(ii) of [46]

(5) By Lemma 3.1, we obtain the relations between $m_{1,2}\mathrm{Cl}(A)$ and $m_{1,2}\mathrm{Int}(A)$.

## 4. *mg*-CLOSED SETS IN BITOPOLOGICAL SPACES

**Definition 4.1** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be

(1) *g-closed* [20] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,

(2) *g$\alpha$-closed* [23] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \alpha(X)$,

(3) *sg-closed* [6] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathrm{SO}(X)$,

(4) *pg-closed* [34] if $\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathrm{PO}(X)$,

(5) *spg-closed* [33] if $\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathrm{SPO}(X)$.

**Definition 4.2** Let $(X, m_X)$ be an $m$-space. A subset $A$ of $X$ is said to be *mg-closed* [33] if $\mathrm{mCl}(A) \subset U$ whenever $A \subset U$ and $U \subset m_X$.

The complement of an $mg$-closed set of $X$ is said to be $m\mathcal{g}$-open. The collection of all $m\mathcal{g}$-open sets is a minimal structure and is denoted by $\mathrm{mCO}(X)$.

**Remark 4.1** Let $(X, \tau)$ be a topological space and $m_X$ an $m$-structure on $X$. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$), then, an $m\mathcal{g}$-closed set is a $\mathcal{g}$-closed (resp. $s\mathcal{g}$-closed, $p\mathcal{g}$-closed, $\mathcal{g}\alpha$-closed, $sp\mathcal{g}$-closed) set.

**Definition 4.3** Let $(X, \tau_1, \tau_2)$ be a bitopological space. A subset $A$ of $X$ is said to be

(1) $(1, 2)^*\mathcal{g}$*-closed* [45] if $\tau_{1,2}\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \subset \tau_{1,2}\mathrm{O}(X)$,

(2) $(1, 2)^*$ $s\mathcal{g}$*-closed* [44] if $(1, 2)^*\mathrm{sCl}(A) \subset U$ whenever $\mathrm{A} \subset U$ and $U \in (1, 2)^*\mathrm{SO}(X)$,

(3) $(1, 2)^*\mathcal{g}\alpha$*-closed* if $(1, 2)^*\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ *and* $U \in (1,2)^*\alpha(\mathrm{X})$,

(4) $(1,2)^*p\mathcal{g}$*-closed* if $(1, 2)^*\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ *and* $U \in (1, 2)^*\mathrm{PO}(X)$,

(5) $(1.2)^*sp\mathcal{g}$*-closed* if $(1, 2)^*\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in (1, 2)^*\mathrm{SPO}(X)$.

**Definition 4.4** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determained by $\tau_1$ and $\tau_2$. A subset $A$ of $X$ is said to be $m(\tau_1, \tau_2)\mathcal{g}$*-closed* (briefly $m_{1,2}\mathcal{g}$-closed) if $A$ is $m\mathcal{g}$-closed in the $m$-space $(X, m(\tau_1, \tau_2))$.

A subset $A$ of $(X, \tau_1, \tau_2)$ is said to be $m(\tau_1, \tau_2)\mathcal{g}$*-open* (breafly $m_{1,2}\mathcal{g}$-open) if $X \setminus A$ is $m(\tau_1, \tau_2)\mathcal{g}$-closed.

**Remark 4.2** Let $(X, \tau_1, \tau_2)$ be a bitopological space.

(1) If $m(\tau_1, \tau_2) = \tau_{1,2}\mathrm{O}(X)$ (resp. $(1, 2)^*\mathrm{SO}(X)$, $(1, 2)^*\mathrm{PO}(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*\mathrm{SPO}(X)$), then an $m(\tau_1, \tau_2)\mathcal{g}$-closed set is $(1, 2)^*\mathcal{g}$-closed (resp. $(1, 2)^*s\mathcal{g}$-closed, $(1, 2)^*pg$-closed, $(1, 2)^*\mathcal{g}\alpha$-closed, $(1, 2)^*sp\mathcal{g}$-closed).

(2) If $m(\tau_1, \tau_2) = \tau_{1,2}\mathrm{O}(X)$(resp. $(1, 2)^*\mathrm{SO}(X)$, $(1, 2)^*\mathrm{PO}(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*\mathrm{SPO}(X)$), then the collection of all $(1, 2)^*\mathcal{g}$-open (resp. $(1, 2)^*s\mathcal{g}$-open, $(1, 2)^*p\mathcal{g}$-open, $(1, 2)^*\mathcal{g}\alpha$-open,

(1,2)*$spg$-open) sets is denoted by $\tau_{1,2}$GO($X$) (resp. (1, 2)*SGO($X$), (1, 2)*PGO($X$), (1, 2)*G$\alpha$($X$), (1, 2)*SPGO($X$)).

(3) The collections $\tau_{1,2}$GO($X$), (1, 2)*SGO($X$), (1, 2)*PGO($X$), (1, 2)*G$\alpha$($X$), (1, 2)*SPGO($X$) are minimal structures on $X$. However, they do not satisfy property $\mathcal{B}$, in general, by Example 2.2 of [44].

**Lemma 4.1** (Noiri [33]). Let (X, mx) be an m-space. For subsets A and B of X, the .following properties hold:

(1) *if $A$ is $m_X$-closed, then $A$ is $mg$-closed,*

(2) *if $m_X$ has property $\mathcal{B}$ and $A$ is $mg$-closed and $m_X$-open, then $A$ is $m_X$-closed,*

(3) *if $A$ is $mg$-closed and $A \subset B \subset$ mCl($A$), then $B$ is $mg$-closed.*

**Theorem 4.1** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. For subsets $A$ and $B$ of $X$, the following properties hold*

(1) *if $A$ is $m_{1,2}$-closed, then $A$ is $m_{1,2}g$-closed,*

(2) *if $A$ is $m_{1,2}g$-closed and $m_{1,2}$-open, then $A$ is $m_{1,2}$-closed,*

(3) *if $A$ is $m_{1,2}g$-closed and $A \subset B \subset m_{1,2}\mathrm{Cl}(A)$, then $B$ is $m_{1,2}g$-closed*

**Proof.** The proof follows from Definition 4.4 and Lemma 4.1.

**Corollary 4.1** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2) = (1, 2)$*SO($X$) For subsets $A$ and $B$ of $X$, the following properties hold:*

(1) *if $A$ is (1, 2)*s-closed, then $A$ is (1, 2)*sg-closed,*

(2) *if $A$ is (1, 2)*sg-closed and (1, 2)*-semi-open, then $A$ is (1, 2)*s-closed,*

(3) *if $A$ is (1, 2)*sg-closed and $A \subset B \subset$ (1, 2)*sCl($A$), then $B$ is (1, 2)*sg-closed* (Theorem 3.2 of [44]).

**Lemma 4.2** (Noiri [33]). *Let $(X, m_X)$ be an m-space, then for each $x \in X$, either $\{x\}$ is $m_X$-closed or $\{x\}$ is $mg$-open.*

**Theorem 4.2** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then for each $x \in X$, either $\{x\}$ is $m_{1,2}$-closed or $\{x\}$ is $m_{1,2}g$-open.*

**Corollary 4.2** (Ravi and Thivagar [44]). *Let $(X, \tau_1, \tau_2)$ be a bitopological space. For each $x \in X$, either $\{x\}$ is $(1, 2)^*$-semi-closed or $\{x\}$ is $(1, 2)^*sg$-open.*

**Lemma 4.3** *A subset $A$ of $X$ is $mg$-open if and only if $F \subset \mathrm{mInt}(A)$ whenever $F \subset A$ and $F$ is $m_X$-closed.*

**Theorem 4.3** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. A subset $A$ of $X$ is $m_{1,2}g$-open if and only if $F \subset m_{1,2}\mathrm{Int}(A)$ whenever $F \subset A$ and $F$ is $m_{1,2}$-closed.*

**Proof.** The proof follows from Definition 4.4 and Lemma 4.3.

**Corollary 4.3** (Ravi and Thivagar [44]). *Let $(X, \tau_1, \tau_2)$ be a bitopological space. A subset $A$ of $X$ is $(1,2)^*sg$-open if and only if $F \subset (1,2)^*\mathrm{sInt}(A)$ whenever $F$ is $(1,2)^*$-semi-closed and $F \subset A$.*

**Lemma 4.4** (Noiri (331). *For subsets $A$ and $B$ of* an *m-space $(X, Mx)$, the following properties hold:*

(1) *if $A$ is m-open, then $A$ is $mg$-open,*

(2) *if $m_X$ has property B and $A$ is $mg$-open and m-closed, then $A$ is m-open,*

(3) *if $A$ is $mg$-open and $\mathrm{mInt}(A) \subset B \subset A$, then $B$ is $mg$-open.*

**Theorem 4.4** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. For subsets $A$ and $B$ of $X$, the following properties hold:*

(1) *if $A$ is $m_{1,2}$-open, then $A$ is $m_{1,2}g$-open,*

(2) *$A$ is $m_{1,2}g$-open and $m_{1,2}$-closed, then $A$ is $m_{1,2}$-open,*

(3) *if $A$ is $m_{1,2}g$-open and $m_{1,2}\mathrm{Int}(A) \subset B \subset A$, then $B$ is $m_{1,2}g$-open.*

**Proof.** The proof folkvws from Definition 4.4 and Lemma 4.4.

**Corollary 4.4** (Ravi and Thivagar [44]). *Let $(X, \tau_1, \tau_2)$ be a bitopological space. For subsets $A$ and $B$ of $X$, the following properties hold:*

(1) *if $A$ is $(1, 2)^*$-semi-open, then $A$ is $(1,2)^*s\mathcal{g}$-open,*

(2) *$A$ is $(1, 2)^*s\mathcal{g}$-open and $(1, 2)^*$-semi-closed, then $A$ is $(1, 2)^*$-semi-open,*

(3) *if $A$ is $(1, 2)^*s\mathcal{g}$-open and $(1, 2)^*\mathrm{sInt}(A) \subset B \subset A$, then $B$ is $(1, 2)^*s\mathcal{g}$-open*

**Lemma 4.5** (Noiri [33]). *Let $(X, m_X)$ be an m-space and $m_X$ have property $\mathcal{B}$ Then, for a subset $A$ of $X$, the following properties are equivalent:*

(1) *$A$ is $m\mathcal{g}$-closed;*

(2) $\mathrm{mCI}(A) \setminus A$ *does not contain, any nonempty m-closed set;*

(3) $\mathrm{mCI}(A) \setminus A$ *is mg-open.*

**Theorem 4.5** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. For subset $A$ of $X$, the following properties are equivalent:*

(1) *$A$ is $m_{1,2}\mathcal{g}$-closed,*

(2) $m_{1,2}\mathrm{Cl}(A) \setminus A$ *does not contain any nonempty $m_{1,2}$-closed set;*

(3) $m_{1,2}\mathrm{Cl}(A) \setminus A$ *is $m_{1,2}\mathcal{g}$-open.*

**Proof.** The proof follows from Definition 4.4 and Lemma 4.5.

**Corollary 4.5** (Ravi and Thivagar [44]). *Let $(X, \tau_1, \tau_2)$ be a bitopological space. For subset $A$ of $X$, the following properties are equivalent:*

(1) *$A$ is $(1, 2)^*s\mathcal{g}$-closed;*

(2) $(1, 2)^*\mathrm{sCI}(A) \setminus A$ *does not contain any nonempty $(1, 2)^*$-semi-closed set;*

(3) $(1, 2)^*\mathrm{sCI}(A) \setminus A$ *is $(1, 2)^*s\mathcal{g}$-open.*

**Lemma 4.6** (Noiri [33]). *Let $(X, m_X)$ be an m-space. A subset $A$ of $X$ is mg-closed if and only if* $\mathrm{mCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and $F$ is m-closed.*

**Theorem 4.6** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. A subset $A$ of $X$ is $m_{1,2}g$-closed if and only if* $m_{1,2}\mathrm{Cl}(A) \cap F = \emptyset$ *whenever $A \cap F = \emptyset$ and $F$ is $m_{1,2}$-closed.*

**Proof.** The proof follows from Definition 4.4 and Lemma 4.6.

**Corollary 4.6** *Let $(X, \tau_1, \tau_2)$ be a bitopological space. For subset $A$ of $X$ is* $(1, 2)^*$*sg-closed if and only if* $(1, 2)^*\mathrm{sCl}(A) \cap F = \emptyset$ *whenever $A \cap F = \emptyset$ and $F$ is* $(1, 2)^*$*-semi-closed*

**Lemma 4.7** *Let $(X, m_X)$ be an m-space, where $m_X$ has property* B. *A subset $A$ of $X$ is mg-closed if and only if* $\mathrm{mCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in \mathrm{mCl}(A)$.

**Theorem 4.7** *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal, structure on $X$ determined by $\tau_1$ and $\tau_2$. A subset $A$ of $X$ is $m_{1,2}g$-closed if and only if* $m_{1,2}\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$ *for every* $x \in m_{1,2}\mathrm{Cl}(A)$.

**Corollary 4.7** *Let $(X, \tau_1, \tau_2)$ be a bitopological space. A subset $A$ of $X$ is* $(1, 2)^*$*sg-closed if and only if* $(1, 2)^*\mathrm{sCl}(\{x\}) \cap A \neq \emptyset$ *for every* $x \in (1, 2)^*\mathrm{sCl}(A)$.

**Definition 4.5** A subset $A$ of an $m$-space $(X, m_X)$ is said to be *locally m-closed* if $A = U \cap F$, where $U \in m_X$ and $F$ is $m_X$-closed.

**Remark 4.3** Let $(X, \tau)$ be a topological space. If $m_X = \tau$ (resp. SO($X$), PO($X$), $\alpha(X)$, SPO($X$)), then a locally $m$-closed set is said to be locally closed [16] (resp. semi-locally closed [47], locally pre-closed [33], $\alpha$-locally closed [17], $\beta$-locally closed [18]).

**Lemma 4.8** (Noiri [33]), *Let $(X, m_X)$ be an m-space and $m_x$ have property* B. *For a subset $A$ of $X$, the following properties are equivalent:*

(1) *$A$ is locally m-closed;*

(2) $A = U \cap \mathrm{mCl}(A)$ *for some* $U \in m_X$;

(3) $\mathrm{mCl}(A) \setminus A$ *is* $m_X$*-closed;*

(4) $A \cup (X \setminus \mathrm{mCl}(A)) \in m_X$;

(5) $A \subset \mathrm{mInt}(A \cup (X \setminus \mathrm{mCl}(A)))$.

**Definition 4.6** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determained by $\tau_1$ and $\tau_2$. A subset $A$ of $X$ is said to be *locally* $m(\tau_1, \tau_2)$*-closed* (briefly locally $m_{1,2}$-closed) if $A$ is locally $m$-closed in the $m$-space $(X, m(\tau_1, \tau_2))$.

Hence, $A$ is locally $m(\tau_1, \tau_2)$-closed if $A = F \cap U$, where $U \in m(\tau_1, \tau_2)$ and $F$ is $m(\tau_1, \tau_2)$-closed.

**Remark 4.4** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$). If a subset $A$ of $X$ is locally $m(\tau_1, \tau_2)$-closed, then $A$ is (1, 2)*-locally closed (resp. (1, 2)*-locally semi-closed, (1, 2)*-locally preclosed, (1, 2)*-locally $\alpha$-closed, (1, 2)*-locally semi-preclosed).

**Theorem 4.8** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *a minimal structure on* $X$ *determined by* $\tau_1$ *and* $\tau_2$. *For a subset* $A$ *of* $X$, *the following properties are equivalent:*

(1) $A$ *is locally* $m(\tau_1, \tau_2)$*-closed;*

(2) $A = U \cap m_{1,2}\mathrm{Cl}(A)$ *for some* $U \in m(\tau_1, \tau_2)$;

(3) $m_{1,2}\mathrm{Cl}(A) \setminus A$ *is* $m_{1,2}$*-closed;*

(4) $A \cup (X \setminus m_{1,2}\mathrm{Cl}(A)) \in m(\tau_1, \tau_2)$;

(5) $A \subset m_{1,2}\mathrm{Int}(A \cup (X \setminus m_{1,2}\mathrm{Cl}(A)))$.

**Proof.** The proof follows from Definition 4.6 and Lenima 4.8.

**Corollary 4.8** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space. For a subset* $A$ *of* $X$, *the following properties are equivalent:*

(1) $A$ *is* (1, 2)**-locally semi-closed;*

(2) $A = U \cap (1, 2)^*\mathrm{sCl}(A)$ *for some* $U \in (1, 2)^*SO(X)$;

(3) $(1. 2)^*\mathrm{sCl}(A) \setminus A$ *is* $(1, 2)^*$-*semi-closed;*

(4) $A \cup (X \setminus (1, 2)^*\mathrm{sCl}(A)) \in (1, 2)^*\mathrm{SO}(X)$;

(5) $A \subset (1, 2)^*\mathrm{sInt}(A \cup (X \setminus (1, 2)^*\mathrm{sCl}(A)))$.

**Lemma 4.9** (Noiri [33]). *Let* $(X, m_X)$ *be an m-space and* $m_X$ *have property* $\mathcal{B}$. *Then a subset* $A$ *of* $X$ *is* $m_X$-*closed if and only if* $A$ *is* $m\mathcal{g}$-*closed and locally m-closed.*

**Theorem 4.9** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $m(\tau_1, \tau_2)$ *a minimal structure on* $X$ *determined by* $\tau_1$ *and* $\tau_2$. *Then a subset* $A$ *of* $X$ *is* $m(\tau_1, \tau_2)$-*closed if and only if* $A$ *is* $m\mathcal{g}_{1,2}$-*closed and locally* $m(\tau_1, \tau_2)$-*closed.*

**Proof.** The proof follows from Definitions 4.5, 4.6 and Lemma 4.9.

**Corollary 4.9** *Let* $(X, \tau_1, \tau_2)$ *be a bitopological space and* $A$ *a subset of* $X$. *Then,*

(1) $A$ *is* $\tau_{1,2}$-*closed if and only if it is* $\tau_{1,2}\mathcal{g}$-*closed and* $(1, 2)^*$-*locally closed,*

(2) $A$ *is* $(1, 2)^*$-*semi-closed if and only if it is* $(1, 2)^*s\mathcal{g}$-*closed and* $(1, 2)^*$-*locally semi-closed,*

(3) $A$ *is* $(1, 2)^*$-*preclosed if and only if it is* $(1, 2)^*$-$p\mathcal{g}$-*closed and* $(1, 2)^*$-*locally preclosed,*

(4) $A$ *is* $(1, 2)^*\alpha$-*closed if and only if it is* $(1, 2)^*\mathcal{g}\alpha$-*closed and* $(1, 2)^*$-*locally* $\alpha$-*closed,*

(5) $A$ *is* $(1, 2)^*$-*semi-preclosed if and only if it is* $(1, 2)^*$-$sp\mathcal{g}$-*closed and* $(1, 2)^*$-*locally semi-preclosed.*

## 5. M-CONTINUITY

**Definition 5.1** Let $(X, m_X)$ and $(Y, m_Y)$ be nonempty sets $X$ and $Y$ with minimal structures $m_X$ and $m_Y$, respectively. a function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* at a point $x \in X$ [40] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$. The function $f$ is said to be *M-continuous* if it has this property at each point $x \in X$.

**Remark 5.1** Let $f : (X, \tau) \to (Y, \sigma)$ be a function.

(1) If $m_X = \tau$ (resp. SO($X$), PO($X$), $\alpha(X)$, SPO($X$)), $m_Y = \sigma$ and $f : (X, m_X) \to (Y, m_Y)$ is $M$-continuous, then $f$ is continuous (resp. semi-continuous [19], precontinuous [25], $\alpha$-continuous [26], semi-precontinuous [4] or $\beta$-continuous [1]).

(2) If $m_X$ = SO($X$) (resp. PO($X$), $\alpha(X)$, SPO($X$)) and $m_Y$ = SO($Y$) (resp. PO($Y$), $\alpha(Y)$, SPO($Y$)), and $f : (X, m_X) \to (Y, m_Y)$ is $M$-continuous, then $f$ is irresolute [13] (resp. preirresolute [27], $\alpha$-irresolute [21], $\beta$-irresolute [28]).

(3) If $m_X = \tau$, $m_Y$ = SO($Y$) (resp. $\alpha(Y)$, $\beta(Y)$), and $f : (X, m_X) \to (Y, m_Y)$ is $M$-continuous, then $f$ is $s$-continuous [9] (resp. strongly $\alpha$-irresolute [21], strongly $\beta$-irresolute [32]).

(4) If $m_X$ = SO($X$), $m_Y = \alpha(Y)$, and $f : (X, m_X) \to (Y, m_Y)$ is $M$-continuous, then $f$ is strongly semi-continuous [37].

**Theorem 5.1** (Noiri and Popa [37]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *$f$ is $M$-continuous at $x \in X$;*

(2) *$x \in \mathrm{mInt}(f^{-1}(V))$ for every $V \in m_Y$ containing $f(x)$;*

(3) *$x \in f^{-1}(\mathrm{mCl}(f(A)))$ for every subset $A$ of $X$ with $x \in \mathrm{mCl}(A)$;*

(4) *$x \in f^{-1}(\mathrm{mCl}(B))$ for every subset $B$ of $Y$ with $x \in \mathrm{mCl}(f^{-1}(B))$;*

(5) *$x \in \mathrm{mInt}(f^{-1}(B))$ for every subset $B$ of $Y$ with $x \in f^{-1}(\mathrm{mInt}(B))$;*

(6) *$x \in f^{-1}(K)$ for every $m_Y$-closed set $K$ of $Y$ such that $x \in \mathrm{mCl}(f^{-1}(K))$.*

For a function $f : (X, m_X) \to (Y, m_Y)$, we define $D_M(f)$ as follows:

$D_M(f) = \{x \in X : f$ is not $M$-continuous at $x\}$.

**Theorem 5.2** (Noiri and Popa [37]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties hold:*

$$\begin{aligned} D_M(f) &= \cup_{G \in m_Y} \{f^{-1}(G) \setminus \mathrm{mInt}(f^{-1}(G))\} \\ &= \cup_{B \in \mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B)) \setminus \mathrm{mInt}(f^{-1}(B))\} \end{aligned}$$

$$= \cup_{B\in\mathcal{P}(Y)} \{\mathrm{mCl}(f^{-1}(B)) \setminus f^{-1}(\mathrm{mCl}(B))\}$$

$$= \cup_{A\in\mathcal{P}(X)} \{\mathrm{mCl}(A) \setminus f^{-1}(\mathrm{mCl}(f(A)))\}$$

$$= \cup_{K\in\mathcal{F}} \{\mathrm{mCl}(f^{-1}(K)) \setminus f^{-1}(K)\},$$

*where $\mathcal{F}$ is the family of $m_Y$-closed sets of $Y$.*

**Theorem 5.3** (Popa and Noiri [40]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *$f$ is M-continuous,*

(2) *$f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ for every $V \in m_Y$;*

(3) *$f(\mathrm{mCl}(A)) \subset \mathrm{mCl}(f(A))$ for every subset $A$ of $X$;*

(4) *$\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{mCl}(B))$ for every subset $B$ of $Y$;*

(5) *$f^{-1}(\mathrm{mInt}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ for every subset $B$ of $Y$;*

(6) *$\mathrm{mCl}(f^{-1}(K)) = f^{-1}(K)$ for every $m_Y$-closed set $K$ of $Y$.*

**Corollary 5.1** *Let $(X, m_X)$ be an m-space and $m_X$ have property $\mathcal{B}$. For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) *$f$ is M-continuous;*

(2) *$f^{-1}(V) \in m_X$ for every $V \in m_Y$;*

(3) *$f^{-1}(F)$ is m-closed in $(X, m_X)$ for every m-closed set $F$ in $(Y, m_Y)$.*

**Definition 5.2** A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *$M^*$-continuous* [29] if $f^{-1}(V) \in m_X$ for every $V \in m_Y$.

**Remark 5.2** (1) If $f : (X, m_X) \to (Y, m_Y)$ is $M^*$-continuous, then $f$ is $M$-continuous. But the converse is not always true as shown in Example 3.4 of [29].

(2) If $m_X$ has property $\mathcal{B}$, then $M$-continuity is equivalent with $M^*$-continuity.

**Definition 5.3** A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *$mg$-continuous* if $f : (X, \mathrm{mGO}(X))$

$\to (Y, m_Y)$ is $M^*$-continuous, equivalently if $f^{-1}(K)$ is $m\mathcal{g}$-closed for each $m$-closed set $K$ of $Y$.

**Definition 5.4** A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *locally mc-continuous* if $f^{-1}(K)$ is locally $m$-closed for every $m$-closed set $K$ of $Y$.

**Theorem 5.4** *A function $f : (X, m_X) \to (Y, m_Y)$, where $m_X$ has property* B, *is M-continuous if and only if f is $m\mathcal{g}$-continuous and locally mc-continuous.*

**Proof.** The proof follows from Definitions 5.3, 5.4 and Lemma 4.9.

**Definition 5.5** A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *contra M-continuous* if $f^{-1}(V)$ is $m$-closed for every $m$-open set $V$ *of* $Y$.

**Theorem 5.5** *If a function $f : (X, m_X) \to (Y, m_Y)$, where $m_X$ has property* B, *is $m\mathcal{g}$-continuous and contra M-continuous, then f is M-continuous.*

**Proof.** Let $V$ be any $m$-open set of $Y$. Since $f$ is $m\mathcal{g}$-continuous, $f^{-1}(V)$ is $m\mathcal{g}$-open. Since $f$ is contra $M$-continuous, $f^{-1}(V)$ is $m$-closed. By Lemma 4.4 $f^{-1}(V)$ $m$-open. By Corollary, 5.1, $f$ is $M$-continuous.

## 6. *M*-CONTINUITY IN BITOPOLOGICAL SPACES

**Definition 6.1** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $\tau_{1,2}$*-continuous*, (1, 2)*-*continuous* [44] or *quasi-continuous* [10] (resp. (1, 2)*-*semi-continuous* [43], (1, 2)*-*precontinuous* [3], (1, 2)*-$\alpha$-*continuous* [3], (1, 2)*-*semi-precontinuous*) if $f^{-1}(V)$ is $\tau_{1,2}$-open (resp. (1, 2)*-semi-open, (1, 2)*-preopen, (1,2)*-$\alpha$-open, (1,2)*-semipreopen) for every $\sigma_{1,2}$-open set $V$ of $Y$.

**Definition 6.2** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (1, 2)*-*semi-irresolute* [3] (resp. (1, 2)*-*preirresolute*, (1, 2)*-$\alpha$-*irresolute* [3], (1, 2)*-*semi-preirresolute*) if $f^{-1}(V)$ is (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-$\alpha$-open, (1, 2)*-semipreopen) of $X$ for every (1, 2)*-semi-open (resp. (1, 2)*-preopen, (1, 2)*-$\alpha$-open, (1, 2)*-semipreopen) of $Y$.

**Definition 6.3** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) is a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$), is said to be *$M(1, 2)$-continuous* (resp. *$M^*$-continuous*) if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ is $M$-continuous (resp. $M^*$-continuous).

**Remark 6.1** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.

(1) If $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$), $m(\sigma_1, \sigma_2) = \sigma_{1,2}O(Y)$ and $f$ is $M(1, 2)$-continuous, then we obtain Definition 6.1.

(2) If $m(\tau_1, \tau_2) = (1, 2)^*SO(X)$ (resp. $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$), $m(\sigma_1, \sigma_2) = (1, 2)^*SO(Y)$ (resp. $(1, 2)^*PO(Y)$, $(1, 2)^*\alpha(Y)$, $(1, 2)^*SPO(Y)$) and $f$ is $M(1, 2)$-continuous, then we obtain Definition 6.2.

By Definition 6.3, Theorem 5.3 and Corollary 5.1, we obtain the following theorem and corollary.

**Theorem 6.1** *Let $(X, \tau_1, \tau_2)$ (resp. $(Y, \sigma_1, \sigma_2)$) be a bitopological space and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure on $X$ (resp. $Y$) determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$) For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *$f$ is $M(1, 2)$-continuous;*

(2) *$f^{-1}(V) = m_{1,2}\mathrm{Int}(f^{-1}(V))$ for every $m_{1,2}$-open set $V$ of $Y$;*

(3) *$f(m_{1,2}\mathrm{Cl}(A)) \subset m_{1,2}\mathrm{Cl}(f(A))$ for every subset $A$ of $X$;*

(4) *$m_{1,2}\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(m_{1,2}\mathrm{Cl}(B))$ for every subset $B$ of $Y$;*

(5) *$f^{-1}(m_{1,2}\mathrm{Int}(B)) \subset m_{1,2}\mathrm{Int}(f^{-1}(B))$ for every subset $B$ of $Y$;*

(6) *$m_{1,2}\mathrm{Cl}(f^{-1}(K)) = f^{-1}(K)$ for every $m_{1,2}$-closed set $K$ of $Y$.*

**Corollary 6.1** *Let $(X, \tau_1, \tau_2)$ (resp. $(Y, \sigma_1, \sigma_2)$) be a bitopological space and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure on $X$ (resp. $Y$) determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *$f$ is $M(1, 2)$-continuous;*

(2) $f^{-1}(V) \in m(\tau_1, \tau_2)$ *for every* $V \in m(\sigma_1, \sigma_2)$;

(3) $f^{-1}(F)$ *is* $m(\tau_1, \tau_2)$*-closed for every* $m(\sigma_1, \sigma_2)$*-closed set* $F$.

By Theorem 6.1 and Corollary 6.1, we obtain the following theorems.

**Theorem 6.2** *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) $f$ *is* (1, 2)**-semi-continuous;*

(2) $f^{-1}(V)$ *is* (1, 2)**-semi-open for each* $\tau_{1,2}$*-open set* $V$ *of* $Y$;

(3) $f^{-1}(K)$ *is* (1, 2)**-semi-closed for each* $\tau_{1,2}$*-closed set* $K$ *of* $Y$;

(4) $(1, 2)^*\mathrm{sCl}(f^{-1}(B)) \subset f^{-1}(\sigma_{1,2}\mathrm{Cl}(B))$ *for every subset* $B$ *of* $Y$;

(5) $f(1, 2)^*\mathrm{sCl}(A)) \subset (1, 2)^*\mathrm{Cl}(f(A))$ *for every subset* $A$ *of* $X$;

(6) $f^{-1}(\sigma_{1,2}\mathrm{Int}(B)) \subset (1, 2)^*\mathrm{sInt}(f^{-1}(B))$ *for every subset* $B$ *of* $Y$.

**Theorem 6.3** *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) $f$ *is* (1, 2)**-semi-irresolute;*

(2) $f^{-1}(V) \in (1, 2)^*\mathrm{SO}(X)$ *for each* $V \in (1, 2)^*\mathrm{SO}(Y)$;

(3) $f^{-1}(K)$ *is* (1, 2)**-semi-closed for each* (1, 2)**-semi-closed set* $K$ *of* $Y$;

(4) $(1, 2)^*\mathrm{sCl}(f^{-1}(B)) \subset f^{-1}((1, 2)^*\mathrm{sCl}(B))$ *for every subset* $B$ *of* $Y$;

(5) $f((1, 2)^*\mathrm{sCl}(A)) \subset (1, 2)^*\mathrm{sCl}(f(A))$ *for every subset* $A$ *of* $X$,

(6) $f^{-1}((1, 2)^*\mathrm{sInt}(B)) \subset (1, 2)^*\mathrm{sInt}(f^{-1}(B))$ *for every subset* $B$ *of* $Y$.

**Remark 6.2** (1) By Theorem 6.2(3), we obtain Remark 2.4 of [44].

(2) By Theorem 6.3(3), we obtain the result established in Theorem 4 of [3].

We denote $D_{M(1,2)}(f) = \{x \in X : f \text{ is not } M(1, 2)\text{-continuous}\}$. Then by Definition 6.3 and Theorem 5.2 we obtain the following theorem.

**Theorem 6.4** *Let* $(X, \tau_1, \tau_2)$ *(resp.* $(Y, \sigma_1, \sigma_2)$*) be a bitopological space and* $m(\tau_1, \tau_2)$ *(resp.*

*$m(\sigma_1, \sigma_2)$) be a minimal structure on $X$ (resp. $Y$) determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). Then, for a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$\begin{aligned} D_{M(1,2)}(f) &= \cup_{G \in m(\sigma_1, \sigma_2)} \{f^{-1}(G) \setminus m_{1,2}\mathrm{Int}(f^{-1}(G))\} \\ &= \cup_{B \in \mathcal{P}(Y)} \{f^{-1}(m_{1,2}\mathrm{Int}(B)) \setminus m_{1,2}\mathrm{Int}(f^{-1}(B))\} \\ &= \cup_{B \in \mathcal{P}(Y)} \{m_{1,2}\mathrm{Cl}(f^{-1}(B)) \setminus f^{-1}(m_{1,2}\mathrm{Cl}(B))\} \\ &= \cup_{A \in \mathcal{P}(X)} \{m_{1,2}\mathrm{Cl}(A) \setminus f^{-1}(m_{1,2}\mathrm{Cl}(f(A)))\} \\ &= \cup_{K \in \mathcal{F}} \{m_{1,2}\mathrm{Cl}(f^{-1}(K)) \setminus f^{-1}(K)\}, \end{aligned}$$

*where $\mathcal{F}$ is the family of $m(\sigma_1, \sigma_2)$-closed sets of $Y$.*

**Definition 6.4** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(1, 2)^*$-*$g$-continuous* [44] (resp. $(1, 2)^*$-*$sg$-continuous* [44], $(1, 2)^*$-*$pg$-continuous*, $(1, 2)^*$-*$\alpha g$-continuous*, $(1, 2)^*$-*$spg$-continuous*) if the inverse image of each $\sigma_{1,2}$-closed set of $Y$ is $(1, 2)^*g$-closed (resp. $(1, 2)^*sg$-closed, $(1, 2)^*pg$-closed, $(1, 2)^*g\alpha$-closed, $(1, 2)^*spg$-closed) in $X$.

**Definition 6.5** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). Then the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *$mg$-continuous* if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ is $mg$-continuous, equivalently if $f^{-1}(V)$ is $m(\tau_1, \tau_2)g$-closed in $X$ for each $\sigma_{1,2}$-closed set $V$ of $Y$.

**Remark 6.3** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. If $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$), $m(\sigma_1, \sigma_2) = \sigma_{1,2}O(Y)$ and $f$ is $mg$-continuous, then by Definition 6.5 we obtain Definition 6.4.

**Definition 6.6** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). Then the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *locally mc-continuous* if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2)$ is locally $mc$-continuous, equivalently, if $f^{-1}(K)$ is locally $m$-closed in $X$ for each $m_{1,2}$-closed set $K$ of $Y$.

**Theorem 6.5** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) is a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$), is $M(1, 2)$-continuous if and only if $f$ is $m\mathcal{g}$-continuous and locally mc-continuous.*

**Corollary 6.2** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then $f$ is*

(1) *$\tau_{1,2}$-continuous if and only if it is $(1, 2)^*\mathcal{g}$-continuous and locally c-continuous,*

(2) *(1, 2)*-semi-continuous if and only if it is (1, 2)*-s$\mathcal{g}$-continuous and locally sc-continuous,*

(3) *(1, 2)*-precontinuous if and only if it is (1, 2)*-p$\mathcal{g}$-continuous and locally pc-continuous,*

(4) *(1, 2)*-$\alpha$-continuous if and only if it is (1, 2)*$\alpha\mathcal{g}$-continuous and locally $\alpha$c-continuous,*

(5) *(1, 2)*-spc-continuous if and only if it is (1, 2)*-sp$\mathcal{g}$-continuous and locally spc-continuous.*

**Definition 6.7** Let If : $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). Then the function $f$ : $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *contra $M$(1, 2)-continuous* if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ is contra $M$-continuous, equivalently if $f^{-1}(V)$ is $m_{1,2}$-closed in $X$ for each $m_{1,2}$-open set $V$ of $Y$.

**Remark 6.4** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. If $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$), $m(\sigma_1, \sigma_2) = \sigma_{1,2}O(Y)$ and $f$ is contra $M$(1, 2)-continuous, then $f$ is contra $\tau_{1,2}$-continuous (resp. contra (1,2)*-semi-continuous, contra (1,2)*-precontinuous, contra (1,2)*-$\alpha$-continuous contra (1, 2)*-semi-precontinuous).

By Definition 6.7 and Theorem 5.5, we obtain the following theorem.

**Theorem 6.6** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) is a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$), is contra $M$(1, 2)-continuous and $m\mathcal{g}$-continuous, then $f$ is $M$(1, 2)-continuous.*

**Corollary 6.3** *Let $f:(X, \tau_1, \tau_2)$ $(Y, \sigma_1, \sigma_2)$ be a function. Then $f$ is*

(1) $\tau_{1,2}$*-continuous if it is* $(1, 2)^*g$*-continuous and contra* $\tau_{1,2}$*-continuous,*

(2) $(1, 2)^*$*-semi-continuous if it is* $(1, 2)^*$*-sg-continuous and contra* $(1, 2)^*$*-semi-continuous,*

(3) $(1, 2)^*$*-precontinuous if it is* $(1, 2)^*$*-pg-continuous and contra* $(1, 2)^*$*-precontinuous,*

(4) $(1, 2)^*$-$\alpha$*-continuous if it is* $(1, 2)^*$-$\alpha g$*-continuous and contra-*$(1, 2)^*$-$\alpha$*-continuous,*

(5) $(1, 2)^*$*-semi-precontinuous if it is* $(1, 2)^*$*-spg-continuous and contra* $(1, 2)^*$*-semi-precontinuous*

## 7. SOME PROPERTIES OF *M*-CONTINUITY FORMS IN BITOPOLOGICAL SPACES

We can obtain some properties of $(1, 2)^*$-continuity forms by using those of $M$-continuity established in [40].

**Definition 7.1** An $m$-space $(X, m_X)$ is said to be $m$-$T_2$ [40] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ containing $x$ and $y$, respectively, such that $U \cap V = \emptyset$.

**Definition 7.2** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ an $m$-structure on $X$ determined by $\tau_1$ and $\tau_2$. The space $(X, \tau_1, \tau_2)$ is said to be $m_{1,2}$-$T_2$ if $(X, m(\tau_1, \tau_2))$ is $m$-$T_2$.

**Remark 7.1** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ an $m$-structure on $X$ determined by $\tau_1$ and $\tau_2$. If $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$) and $X$ is $m_{1,2}$-$T_2$, then $X$ is $\tau_{1,2}$-$T_2$ (resp. $(1, 2)^*$-$sT_2$, $(1, 2)^*$-$pT_2$ [38], $(1, 2)^*$-$\alpha T_2$, $(1, 2)^*$-$spT_2$).

**Lemma 7.1** (Popa and Noiri [40]). *If $f : (X, m_X) \to (Y, m_Y)$ is an M-continuous injection and $(Y, m_Y)$ is $m$-$T_2$, then $(X, m_X)$ is $m$-$T_2$.*

**Theorem 7.1** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, where $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) is a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). If $f$ is an $M(1, 2)$-continuous injection and $(Y, \sigma_1, \sigma_2)$ is $m_{1,2}$-$T_2$, then $(X, \tau_1, \tau_2)$ is $m_{1,2}$-$T_2$.*

**Proof.** The proof follows from Definition 7.2 and Lemma 7.1.

**Corollary 7.1** *Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $\tau_{1,2}$-continuous (resp. (1, 2)*-semi-continuous, (1, 2)*-precontinuous, (1, 2)*-$\alpha$-continuous, (1, 2)*-semi-precontinuous) injection and $(Y, \sigma_1, \sigma_2)$ is (1, 2)*-$T_2$, then $(X, \tau_1, \tau_2)$ is (1, 2)*-$T_2$ (resp. (1, 2)*-$sT_2$, (1, 2)*-$pT_2$, (1, 2)*-$\alpha T_2$, (1, 2)*-$spT_2$).*

**Corollary 7.2** *If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1, 2)*-semi-irresolute (resp. (1, 2)*-preirresolute, (1, 2)*-$\alpha$-irresolute, (1, 2)*-semi-preirresolute) injection and $(Y, \sigma_1, \sigma_2)$ is (1, 2)*-$sT_2$ (resp (1, 2)*-$pT_2$, (1, 2)*-$\alpha T_2$, (1, 2)*-$spT_2$), then $(X, \tau_1, \tau_2)$ is (1, 2)*-$sT_2$ (resp. (1, 2)*-$pT_2$, (1, 2)*-$\alpha T_2$, (1, 2)*-$spT_2$).*

**Definition 7.3** Let $(X, m_X)$ be an $m$-space and $K$ a subset of $X$.

(1) $K$ is said to be *m-compact* [40] if every cover of $K$ by subsets of $m_X$ has a finite subcover,

(2) $(X, m_X)$ is said to be *m-compact* [40] if every cover of $X$ by subsets of $m_X$ has a finite subcover.

**Definition 7.4** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$.

(1) A subset $K$ of $X$ is said to be *(1, 2)*m-compact* if $K$ is $m$-compact in $(X, m(\tau_1, \tau_2))$,

(2) $(X, \tau_1, \tau_2)$ is said to be *(1, 2)*m-compact* if $(X, m(\tau_1, \tau_2))$ is $m$-compact.

**Remark 7.2** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ an $m$-structure on $X$ determined by $\tau_1$ and $\tau_2$. If $m(\tau_1, \tau_2) = \tau_{12}O(X)$ (resp. (1, 2)*SO($X$), (1, 2)*PO($X$), (1, 2)*$\alpha(X)$, (1, 2)*SPO($X$)). If $X$ is (1, 2)*$m$-compact, then $X$ is (1, 2)*-compact (resp. (1, 2)*-semicompact, (1, 2)*-precompact, (1, 2)*-$\alpha$-compact, (1, 2)*-semi-precompact).

**Lemma 7.2** (Popa and Noiri [40]). *Let $f: (X, m_X) \to (Y, m_Y)$ be an M-continuous function Then the following properties hold:*

(1) *If a subset $K$ of $X$ is m-compact, then $f(K)$ is m-compact.*

(2) *If $f$ is surjective and $(X, m_X)$ is m-compact, then $(Y, m_Y)$ is m-compact.*

**Theorem 7.2** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$) and $f$ be $M(1, 2)$-continuous. Then the following properties hold:*

(1) *If a subset $K$ of $X$ is $m(\tau_1, \tau_2)$-compact, then $f(K)$ is $m(\sigma_1, \sigma_2)$-compact.*

(2) *If $f$ is surjective and $(X, \tau_1, \tau_2)$ is $m(\tau_1, \tau_2)$-compact, then $(Y, \sigma_1, \sigma_2)$ is $m(\sigma_1, \sigma_2)$-compact.*

**Proof.** The proof follows from Definition 7.4 and Lemma 7.2.

**Corollary 7.3** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $(1, 2)^*$-continuous (resp. $(1, 2)^*$-semi-continuous, $(1, 2)^*$-precontinuous, $(1, 2)^*$-$\alpha$-continuous, $(1, 2)^*$-semi-precontinuous) function.*

(1) *If a subset $K$ of $X$ is $(1, 2)^*$-compact (resp. $(1, 2)^*$-semicompact, $(1, 2)^*$-precompact, $(1, 2)^*$-$\alpha$-compact, $(1, 2)^*$-semi-precompact), then $f(K)$ is $(1, 2)^*$-compact in $Y$.*

(2) *If $f$ is surjective and $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-compact (resp. $(1, 2)^*$-semicompact, $(1, 2)^*$-precompact, $(1, 2)^*$-$\alpha$-compact, $(1, 2)^*$-semi-precompact), then $(Y, \sigma_1, \sigma_2)$ is $(1, 2)^*$-compact in $Y$.*

**Corollary 7.4** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(1, 2)^*$-semi-irresolute (resp. $(1, 2)^*$-preirresolute, $(1, 2)^*$-$\alpha$-irresolute, $(1, 2)^*$-semi-preirresolute) function.*

(1) *If a subset $K$ of $X$ is $(1, 2)^*$-semicompact (resp. $(1, 2)^*$-precompact, $(1, 2)^*$-$\alpha$-compact, $(1, 2)^*$-semi-precompact), then $f(K)$ is $(1, 2)^*$-semicompact (resp. $(1, 2)^*$-precompact, $(1, 2)^*$-$\alpha$-compact, $(1, 2)^*$-semi-precompact) in $Y$.*

(2) *If $f$ is surjective and $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-semicompact (resp. $(1, 2)^*$-precompact, $(1, 2)^*$-$\alpha$-compact, $(1, 2)^*$-semi-precompact), then $(Y, \sigma_1, \sigma_2)$ is $(1, 2)^*$-semicompact (resp. $(1, 2)^*$-precompact, $(1, 2)^*$-$\alpha$-compact, $(1, 2)^*$-semi-precompact).*

**Definition 7.5** An $m$-space $(X, m_X)$ is said to be *m-connected* [40] if $X$ can not be written as the union of two nonempty disjoint $m$-open sets of $X$.

**Definition 7.6** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $m(\tau_1, \tau_2)$ be a minimal structure on $X$ determined by $\tau_1$ and $\tau_2$. Then $(X, \tau_1, \tau_2)$ is said to be *$m(\tau_1, \tau_2)$-connected* if $(X, m(\tau_1, \tau_2))$ is $m$-connected.

**Remark 7.3** Let $(X, \tau_1, \tau_2)$ be a bitopological space, $m(\tau_1, \tau_2)$ an $m$-structure on $X$ determined by $\tau_1$ and $\tau_2$ and $m(\tau_1, \tau_2) = \tau_{1,2}O(X)$ (resp. $(1, 2)^*SO(X)$, $(1, 2)^*PO(X)$, $(1, 2)^*\alpha(X)$, $(1, 2)^*SPO(X)$). If $(X, \tau_1, \tau_2)$ is $m(\tau_1, \tau_2)$-connected, then $X$ is (1, 2)*-connected (resp. (1, 2)*-semi-connected, (1, 2)*-preconnected, (1, 2)*-$\alpha$-connected, (1, 2)*-semi-preconnected).

**Lemma 7.3** (Popa and Noiri [40]). *Let $(X, m_X)$ be m-connected, where $m_X$ has property* $\mathcal{B}$, *and $f : (X, m_X) \to (Y, m_Y)$ is an M-continuous surjection, then $(Y, m_Y)$ is m-connected.*

**Theorem 7.3** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). If $f$ is an $M(1, 2)$-continuous surjection and $(X, \tau_1, \tau_2)$ is $m(\tau_1, \tau_2)$-connected, then $(Y, \sigma_1, \sigma_2)$ is $m(\sigma_1, \sigma_2)$-connected.*

**Proof.** The proof follows from Definition 7.6 and Lemma 7.3 since $m(\tau_1, \tau_2)$ has property $\mathcal{B}$.

**Corollary 7.5** *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (1, 2)*-continuous (resp. (1, 2)*-semi-continuous, (1, 2)*-precontinuous, (1, 2)*-$\alpha$-continuous, (1, 2)*-semi-precontinuous) surjection and $(X, \tau_1, \tau_2)$ is (1, 2)*-connected (resp. (1, 2)*-semi-connected, (1, 2)*-preconnected, (1, 2)*-$\alpha$-connected, (1, 2)*-semi-preconnected), then $(Y, \sigma_1, \sigma_2)$ is (1, 2)*-connected in Y.*

**Proof.** The proof follows from Theorem 7.3.

**Corollary 7.6** *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (1, 2)*-semi-irresolute (resp. (1, 2)*-preirresolute, (1, 2)*-$\alpha$-irresolute, (1, 2)*-semi-preirresolute) surjection and $(X, \tau_1, \tau_2)$ is (1, 2)*-semi-connected (resp. (1, 2)*-preconnected, (1, 2)*-$\alpha$-connected, (1, 2)*-semi-preconnected), then $(Y, \sigma_1, \sigma_2)$ is (1, 2)*-semi-connected (resp. (1, 2)*-preconnected, (1, 2)*-$\alpha$-connected, (1, 2)*-semi-preconnected).*

**Definition 7.7** A function $f : (X, m_X) \to (Y, m_Y)$ is said to have a *strongly m-closed graph*

(resp. *m-closed graph*) [40] if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in m_X$ containing $x$ and $V \in m_Y$ containing $y$ such that $[U \times \mathrm{mCl}(V)] \cap G(f) = \emptyset$. (resp. $[U \times V]\ G(f) = \emptyset$).

**Definition 7.8** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). Then the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have a *strongly* $m(\tau_1, \tau_2)$*-closed graph* (resp. $m(\tau_1, \tau_2)$*-closed graph*) if $f : (X, m(\tau_1, \tau_2)) \to (Y, m(\sigma_1, \sigma_2))$ has a strongly $m$-closed graph (resp. $m$-closed graph).

**Lemma 7.4** (Popa and Noiri [40]). *Let* $f : (X, m_X) \to (Y, m_Y)$ *be a function.*

(1) *if $f$ is M-continuous and $(Y, m_Y)$ is $m$-$T_2$, then $G(f)$ is strongly m-closed,*

(2) *if $f$ is a surjection with a strongly m-closed graph, then $(Y, m_Y)$ is $m$-$T_2$,*

(3) *if $m_X$ has property $\mathcal{B}$ and $f$ is an M-continuous injection with an m-closed graph, then $(X, m_X)$ is $m$-$T_2$.*

**Theorem 7.4** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $m(\tau_1, \tau_2)$ (resp. $m(\sigma_1, \sigma_2)$) be a minimal structure determined by $\tau_1$ and $\tau_2$ (resp. $\sigma_1$ and $\sigma_2$). Then the following properties hold:*

(1) *if $f$ is M-continuous and $(Y, \sigma_1, \sigma_2)$ is $m_{1,2}$-$T_2$, then $G(f)$ is strongly $m(\tau_1, \tau_2)$-closed,*

(2) *if $f$ is a surjection with a strongly $m(\tau_1, \tau_2)$-closed graph, then $(Y, \sigma_1, \sigma_2)$ is $m_{1,2}$-$T_2$,*

(3) *if $f$ is an $M(1, 2)$-continuous injection with an $m(\tau_1, \tau_2)$-closed graph, then $(X, \tau_1, \tau_2)$ is $m_{1,2}$-$T_2$.*

**Remark 7.4** (1) As in cases of Theorems 7.1, 7.2 and 7.3, we obtain two corollaries to Theorem 7.4.

(2) Other results for $M(1, 2)$-continuous functions in bitopological spaces follow from Theorems 4.3, 4.4 and 4.5 of [40].

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